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19, rue Auber — Paris (9e)
1. - Introduction

The simultaneous observations of directions to satellites from two or more stations enables the directions of the station-to-station vectors, oriented in an astronomical reference system, to be determined. The distribution of the stations is, however, often such that intervisibility between only two stations at a time is possible. The resulting space directions joining the pairs of stations provide the basis of two three-dimensional spatial triangulations. The Smithsonian Astrophysical Observatory's network of Baker-Nunn cameras is typical of this situation.

The accuracy with which the station-to-station vectors can be determined is, of course, dependent on the accuracy of the observed directions, on the rigor of the variety of "corrections" that must of necessity be applied to the observed quantities, and on the geometry of the solution. The last of these factors will be investigated here, assuming that the covariance matrix of the observations is known and that the corrections have been properly applied.

In such an analysis it is not sufficient to treat only those parameters defining the relative station-satellite positions for any pair of stations; it is also necessary to consider those defining the frequency with which the satellite can be observed. For, obviously, such an optimization is of little value if it leads to conditions that are only infrequently satisfied by a particular combination of stations and satellites.

Some earlier results of such a study have been carried out in [1], and the results of a similar analysis for different combinations of simultaneous range and direction observations have been presented in [2].

2. - A priori accuracy estimates of the station-to-station vector

If the $i$th pair of simultaneously observed station-to-satellite unit vectors is denoted by $\mathbf{u}_i$ and $\hat{\mathbf{u}}_i$, and the station-to-station unit vector by $\hat{\mathbf{u}}_i$, the condition that these vectors must satisfy can be written as
Expressing this relation in terms of the observed quantities $\delta_{1,2}$ (declination) and $\alpha_{1,2}$ [cosine $\delta_{1} \times$ (right ascension − sidereal time)] and of the initial values $\delta_{3}$ and $\alpha_{3}$ for the station-to-station vector, and linearizing, gives

$$
\sum_{p=1}^{3} \left( a_{p}^{i} \delta_{p}^{i} + b_{p}^{i} \alpha_{p}^{i} \right) + \Delta^{i} = 0, \quad i = 1 \ldots n.
$$

The $a_{p}^{i}$, $b_{p}^{i}$, and $\Delta^{i}$ are functions of the $\delta_{1,2,3}$ and $\alpha_{1,2,3}$; the $\delta_{1,2}$ and $\alpha_{1,2}$ are corrections to the observations making the $i$th pair; and the $\delta_{3}$ and $\alpha_{3}$ are the sought corrections to the $\delta_{3}$ and $\alpha_{3}$.

Application of the law of propagation of variances leads to an estimate of the covariance matrix of the $\delta_{3}$ and $\alpha_{3}$.

If all $n$ observations are assumed to be uncorrelated and to have equal variances $\sigma_{s}^{2}$ and if there is no correlation between the observed $\delta^{i}$ and $\alpha^{i}$ for any one observation, then the covariance matrix of the adjusted station-to-station vector becomes

$$
\sigma^{2} \left( \begin{array}{c} \delta_{3} \\ \alpha_{3} \end{array} \right) = \sigma_{s}^{2} \left( \begin{array}{cc} \sum a_{i}^{2} A_{i} a_{3}^{i} & \sum a_{i}^{2} A_{i} b_{3}^{i} \\ \sum b_{i}^{2} A_{i} a_{3}^{i} & \sum b_{i}^{2} A_{i} b_{3}^{i} \end{array} \right)
$$

with

$$
A_{i} = \left[ \left( a_{i}^{i} \right)^{2} + \left( b_{i}^{i} \right)^{2} + \left( a_{i}^{2} \right)^{2} + \left( b_{i}^{2} \right)^{2} \right]^{-1}
$$

Thus, for the above assumptions, the covariance matrix is simply the inverse of the sum of the weight-function matrices of the individual pairs of simultaneous observations.

The above matrix can be readily projected onto a plane normal to the station-to-station vector; two directions in this projection that are of particular interest are the direction along the intersection of this "normal" plane with the plane containing the two stations and the earth's center of mass and the direction at right angles to this first one. The latter component will approximate the
POSITION DETERMINATION FROM SIMULTANEOUS ....

accuracy in azimuth of the station—to—station vector, while the former is readily converted into an accuracy estimate of relative height.

The weight functions \( a_3^i A^i a_3^j \), \( a_3^i b_3^j \), \( b_3^i A^i b_3^j \) are readily evaluated for different geometrical station—satellite configurations. For this evaluation a local coordinate system and some auxiliary quantities are used to keep the number of variables to a minimum. These quantities are (see Fig. 1).

\( Z \) — The angle between the plane containing the two stations and the satellite (the “S plane”) and the plane containing the two stations and the earth’s mass center (the “V plane”).

\( L \) — The straight—line distance between the two stations.

\( H \) — The height of the satellite above the plane containing the two stations and normal to the V plane (the “A plane”).

\( C \) — The shortest distance of the satellite position from a plane normal to both the V and the A planes and passing through the earth’s center of mass (the “C plane”).

The position of the satellite relative to the two stations can then be specified by the quantities \( Z, L/H, \) and \( C \) without the necessity of considering the effect of the earth’s curvature on the geometry.

The local coordinate system is defined by one axis along the station—to—station vector and one axis in the V plane.

In this system the \( a_3^i A^i a_3^j \) is equivalent to the weight contribution of the \( i \)th observation to the vertical component of the station—to—station vector \( W^V \), the \( b_3^i A^i b_3^j \) is equivalent to the weight contribution in azimuth \( W^A \), and the \( a_3^i A^i b_3^j \) is a measure of the correlation between these two components.

Figures 2, 2a, and 2b give some typical results for the coefficients \( W^V , W^A \), and \( |W^V_A| \) for the case \( C = 0 \). The sign of \( W^V_A \) will depend on which side of the V plane the satellite is located. Figure 3 gives more general results for \( W^V , W^A \), and \( L/H = 1 \). Results for other \( L/H \) values are given in [1].

Some conclusions can immediately be drawn from these results:

(i) For observations symmetrically distributed about the V plane the correlation between the azimuth and the height will be zero.

(ii) The greater the ratio \( L/H \) the greater will be the \( W^V \) and \( W^A \). For example, for \( C = 0 \) and \( L/H \to \infty \) both \( W^V \) and \( W^H \) \( \to 2 \), as would be expected since this is equivalent to two direct measurements of the station—to—station vector. But intervisibility requirements obviously impose some restrictions on the maximum value for \( L/H \).

(iii) For any given value of \( L/H \) and \( Z \) the \( W^V \) and \( W^A \) are a maximum for \( C = 0 \).
The azimuth component will nearly always be determined with a higher accuracy than will the vertical component unless special selection criteria with respect to the geometrical distribution are used. For any pair of observations \( W_v \) will equal \( W_A \) only when \( Z = 45^\circ \). Certain other groupings of observations will also yield equal accuracies in the two components [1].

The common visibility area of the two stations is defined by the maximum zenith distance \( z_{\text{max}} \) at which the observations can be made, and, because of the earth's curvature, by both \( L \) and the satellite height \( h \) above the earth's surface. These common visibility regions can be projected onto the curves of equal \( W_v \) and \( W_A \) given in Fig. 3, and average values \( \langle W_v, W_A \rangle \) for these weight functions can be estimated.

Tables 1 and 1a give results for \( z_{\text{max}} = 75^\circ \) for \( 500 < H < 4000 \) and for \( 500 < L < 6000 \).

Over the ranges of \( L \) and \( H \) considered, these average weight functions show only a slight dependence on \( H \), so that the ambiguity inherent in the definition of \( H \) is immaterial and this quantity can be replaced by \( h \). Furthermore, the \( W_v \) and \( W_A \) show a marked linear dependency on \( L/h \) (or \( L/H \)) (see Fig. 4), which can be expressed by

\[
\begin{align*}
W_A &= 0.34 \left( \frac{L}{h} \right) - 0.15 \\
W_v &= 0.09 \left( \frac{L}{h} \right) - 0.03
\end{align*}
\]

Consequently, for \( n \) pairs of simultaneous observations of a satellite with average height \( L \) and uniformly distributed through the common coverage area, the accuracy of the station-to-station vector determination can be written as

\[
\begin{bmatrix}
\sigma_V^2 & \sigma_{VA}^2 \\
\sigma_{AV}^2 & \sigma_A^2
\end{bmatrix}
= \frac{1}{n} \begin{bmatrix}
1 & 0 \\
0.09 (L/h) - 0.03 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0.34 (L/h) - 0.15 & 1
\end{bmatrix}
\]

For \( \sigma_V^2 = \sigma_A^2 \) and \( \sigma_{VA}^2 = 0 \), the observations must be distributed uniformly through certain regions of the common coverage area. For convenience, these regions will be called the partial common coverage area. Their construction is discussed in [1], and the accuracy of the resulting station-to-station vector can be written as
3. Optimization of Parameters

The formulations (1) and (2) indicate the importance of using as large a value as is practically feasible for the $L/h$ ratio, if the number of necessary film reductions is to be minimized. But situations may occur in which the time element is of greater importance than the total number of observations. Now, in a specified time interval, the number of observations $n$ that can be made to a particular object will be proportional to the probability $p$ of the satellite being simultaneously visible from the two stations. Thus, the optimum combination of parameters is the one for which

$$\frac{1}{p} = \begin{bmatrix} 1/W_V & 0 \\ 1/\bar{p} \\ 0 & 1/W_A \end{bmatrix}$$

is a minimum.

The probability $p$ is a function of numerous variables, including those defining the satellite orbit, the station locations, and the position of the earth in its orbit about the sun. But, fortunately, many of these parameters can be eliminated if average values $\bar{p}$ are used. For example, for latitudes up to about $50^\circ$ and for high-inclination orbits ($60^\circ < i < 120^\circ$) the probability of the satellite being in the common coverage area is—to a good approximation—proportional to the size of this area. And as observations are usually made over time periods of several months, the factors causing variations in the probability of the object being brighter than the sky background can also be averaged. Then $\bar{p}$ becomes a function only of $L$, $h$, $z_{\text{max}}$ and of whether the observations are made to an active or to a passive object. Figures 5 and 5a evaluate $\bar{p}$ for $z_{\text{max}} = 75^\circ$ and variable $L$ and $h$. The sun is assumed to be at least $18^\circ$ below the horizon (≡ to astronomical twilight) for the object to be visible. Figure 5 refers to observations distributed throughout the entire common coverage area of the two stations, while Fig. 5a refers to observations confined to the partial common coverage areas. Weather conditions and instrumental or other failures are, of course, ignored.

These probabilities can be interpreted as the average percentage of time that the satellite can be expected to be visible from both stations.

The quantities $\bar{p}$, $W$ can now be evaluated; the results are presented in
Fig. 6. For either active or passive objects, and for observations distributed through the entire common overlap area, both $\mathbf{p} \cdot \mathbf{W}_A$ and $\mathbf{p} \cdot \mathbf{W}_V$ are a maximum for the same combination of $L$ and $h$. The relationship can be approximated by

$$L = 0.9h + 1.6 . \quad (L, h \text{ in Mm})$$

For observations distributed throughout the partial coverage area, this relation can be approximated by

$$L = 0.7h + 1.3 .$$

Now, depending on whether the period of observing is of greater interest than the total number of observations required to obtain a certain a priori accuracy and on whether it is of greater interest to obtain equal accuracy in height and azimuth or to obtain a higher accuracy in azimuth than in height, the above results can be used to optimize the values $h$ and $L$ accordingly (see the following example).

The total observing time required to obtain the preassigned accuracy of the station-to-station vector can also be determined from the probability curves given in Figs. 5 and 5a. As mentioned before, these probabilities represent the percentage of time in which the satellite can be observed simultaneously from the two stations, ignoring losses due to weather and to malfunctioning of equipment. Thus, if $\beta_1$ and $\beta_2$ are the loss factors at the two stations due to these latter sources and if $\Delta t$ is the minimum time interval permissible (in minutes) between successive observations in a pass, the number $N$ of simultaneous observations that can be made to an object is given by

$$N = \frac{1440 \times p \times \beta_1 \beta_2}{\Delta t} \text{ obs/day} .$$

Example

Consider the Baker-Nunn cameras at San Fernando (station 9004) and Athens (station 9091). The chord distance between these stations is about 2600 km, and their mean latitude is about 35° North. The two satellites to be considered for simultaneous observations are 6605601 (Pageos) and 6800201 (Geos 2). Their respective average heights are 4000 and 1300 km, and both are in near-polar orbits.

The accuracy of a synthetic observation—derived from a sequence of seven observations—is assumed to be 1°5. The required accuracy for the station-to-station vector is taken as 0°5 or better in any component. For Pageos, $L/h = 0.65$, and for Geos 2, $L/h = 2.0$. Geos 2 is observed in the flash mode only.

By use of Eq. (1), the average weight factors and hence the number of observations...
observations required \( n \) can be determined (rows 2 and 3, Table 2). The probabilities \( p \) (row 4) of observing the satellite simultaneously from the two stations are determined from Figs. 5 and 5a, and the number of observations \( N \) that can be made per day can be computed from Eq. (3). Typical figures of \( \beta_1 = \beta_2 = 50^{\circ}/1, \Delta t = 4 \text{ min}, \) are used. The total observing time required is given by \( n/N. \)

The results in Table 2 indicate that, for observations distributed through the entire overlap, the same accuracies are attainable for either satellite if the observation period is the same. However, almost five times as many Pageos observations are required as Geos observations. If the duration of the observing period is not of primary importance, but if the number of observations is, then it may be preferable to observe Geos 2 only when it occurs in the partial common coverage areas. For now, only about 30 observations are required to ensure an accuracy of \( 0'\,5 \) in all components of the station-to-station vector. Otherwise, for observations throughout the common coverage area, about 60 observations are required to ensure that the accuracy of the vertical component is of the order of \( 0'\,5. \) But the azimuth component will, in this case, be determined with a higher accuracy, about \( 0'\,3. \)

Acknowledgment

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References


### Table 1

<table>
<thead>
<tr>
<th>L/H</th>
<th>H</th>
<th>500 km</th>
<th>1000 km</th>
<th>2000 km</th>
<th>3000 km</th>
<th>4000 km</th>
<th>Average</th>
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<td>500 km</td>
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<td>0.086</td>
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<tr>
<td></td>
<td>1000 km</td>
<td>0.152</td>
<td>0.175</td>
<td>0.195</td>
<td>0.210</td>
<td>0.220</td>
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<tr>
<td></td>
<td>2000 km</td>
<td>0.281</td>
<td>0.311</td>
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<td></td>
<td>3000 km</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Average</td>
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<td>0.09</td>
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### Table 1a

<table>
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<th>H</th>
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<th>3000 km</th>
<th>4000 km</th>
<th>Average</th>
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<td></td>
<td>2000 km</td>
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<td>0.098</td>
<td>0.100</td>
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<td></td>
<td>4000 km</td>
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</tr>
<tr>
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### Table 2

<table>
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<th>Geos 2 (active)</th>
<th>Comments</th>
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<td>1</td>
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<td>0.07</td>
<td>0.53</td>
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<td></td>
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<td>0.03</td>
<td>0.15</td>
<td>From Eq. (2)</td>
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<td></td>
</tr>
<tr>
<td>3</td>
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<td>270</td>
<td>60</td>
<td>Number of observations required</td>
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<td>17</td>
<td></td>
</tr>
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<td></td>
<td></td>
<td>147</td>
<td>30</td>
<td>for $\sigma \leq 0.5$</td>
</tr>
<tr>
<td>4</td>
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<td>0.0076</td>
<td>From Fig. 5</td>
</tr>
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<td>0.022</td>
<td>0.0028</td>
<td>From Fig. 5a</td>
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<td>4.05</td>
<td>0.66</td>
<td>Equation with $\rho_1 = \rho_2 = 50%$</td>
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<tr>
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<td>0.26</td>
<td>$\Delta t = 4$ min</td>
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<td>6</td>
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<td>67</td>
<td>88</td>
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<td>$n_{E}/N_{E}$</td>
<td>75</td>
<td>110</td>
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---

From Eq. 4.05

$\rho_1 = \rho_2 = 50\%$

$\Delta t = 4$ min

Number of observations required

Number of days

Number of days

Number of days
Figure 1 - Station-satellite configuration defined in terms of L, H, C, and Z. L is the distance between the two stations, H the height of the satellite above the A plane, C the shortest distance between the satellite and the C plane, and Z the angle between the V and S planes.
Figure 2 - Weight component in the vertical plane.

Figure 2a - Correlation terms between the components in the vertical plane and in azimuth.

Figure 2b - Weight component in azimuth as a function of the satellite-plane angle $Z$ and the ratio $L/H$. 

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Figure 3 - $W_V$ and $W_A$ as functions of $Z$, $L/H$, and $C_P$.

$$W_A = 0.34 \left( \frac{L}{h} \right) - 0.15$$

$$W_V = 0.09 \left( \frac{L}{h} \right) - 0.03$$

$$W_E = 0.17 \left( \frac{L}{h} \right) - 0.05$$

Figure 4 - Relation between average weight functions and $L/h$. 
Figure 5 - Average probability of satellite being simultaneously visible from two stations.

Figure 5a - Average probability of satellite being visible in the partial common coverage area.
Figure 6 - Evaluation of $\bar{W}/\bar{p}$ as a function of $L$ and $h$ for active and passive satellites.