Postglacial rebound and sea level contributions to changes in the geoid and the Earth’s rotation axis

P. Johnston and K. Lambeck
Research School of Earth Sciences, Australian National University, Canberra ACT 0200, Australia. E-mail: paul@rses.anu.edu.au

Accepted 1998 September 16. Received 1998 September 11; in original form 1998 March 12

SUMMARY
The rate of change of the spherical harmonic degree 2 component of the Earth’s gravitational potential \( C_{20} \) and polar wander velocity are two signals which are sensitive to Late Pleistocene deglaciation, current changes in sea level and deep mantle viscosity. Different load and earth models have been used in earlier papers to predict the component of these geophysical signals caused by the collapse of the last great ice sheets and recent melting of polar ice caps. In this paper, we present a systematic analysis of the dependence of the predictions on parameters of the ice and earth model. We show that the key parameters of the ice model which govern the predictions are the mass, the location of the centre of mass and the midpoint of the deglaciation phase. Of secondary importance is the length of the deglaciation phase and the mean ice load prior to the Last Glacial Maximum. These conclusions enable us to make a more robust inference of mantle viscosity than has been made before, allowing for the uncertainties in the model of Late Pleistocene and present deglaciation. As previous authors have shown, the lower-mantle viscosity is the most important rheological parameter and therefore the \( C_{20} \) observation complements sea level observations, which are primarily sensitive to lithospheric thickness and the viscosity of the upper part of the mantle. Using realistic constraints on the sizes, locations and timing of deglaciation of the Late Pleistocene ice sheets and current changes in polar ice caps, the observation of \( C_{20} \) is used to infer lower-mantle viscosity as a function of the present rate of sea level change. If the present rate of non-steric sea level change is 1 mm yr\(^{-1}\) and that change has been occurring for less than 1000 years, then the lower-mantle viscosity satisfies \( \log_{10} \eta_{lm} = 21.82 \pm 0.15 \), which is consistent with inferences drawn from recent sea level analyses and confirms other analyses of the \( C_{20} \) observation. If the polar wander signal is produced entirely by postglacial rebound and current sea level change, no more than 20 per cent of the present contribution to global sea level change comes from Greenland. The above conclusions also hold if the density discontinuities at 420 and 670 km are modelled as phase boundaries rather than material (chemical) boundaries.

Key words: Earth’s rotation, geoid, glacial rebound, mantle viscosity, true polar wander.

1 INTRODUCTION
The Earth’s past glacial cycles have left their traces in a number of geological, geomorphological and geodetic observations either directly as glacial signatures within formerly glaciated regions or less directly as evidence of fluctuating sea levels approximately in phase with the waxing and waning of the ice sheets. The Earth’s rotation and the motions of low-orbit satellites are also influenced by the past existence of these ice sheets because of the global nature of the planet’s deformation. It is the delayed response of the planet to surface load changes that occurred primarily more than 10,000 years ago that makes it possible to observe this phenomenon in the geological record as well as with modern instrumentation. The various signatures provide a measure of the Earth’s response to changes in surface loading on timescales of thousands of years and as such provide important evidence for the viscosity of the mantle. However, they also contain information on the history of the former ice sheets, particularly for the final decay stage of the last Late Pleistocene deglaciation, and the analysis of any particular set of observations is often limited by the degree to which parameters describing the Earth response and the ice sheets can be separated. The Earth’s rotational irregularities or the satellite orbit perturbations are particularly important...
in this regard because they are global, long-wavelength responses of the Earth, whereas some of the other observations reflect shorter-wavelength and regional components. The long-wavelength responses can be anticipated to be indicative of the rheological properties deep in the mantle as well as the bulk properties of the load. The shorter-wavelength responses, on the other hand, reflect shallower properties of the Earth and the more detailed distributions of the ice through space and time. An ideal formulation of the glacial rebound phenomenon takes advantage of this complementarity and combines the two classes of observations.

The importance of the long-wavelength information is reflected in the numerous papers published within the past two decades on different aspects of the analysis of the satellite and rotational data (O'Connell 1971; Nakiboglu & Lambeck 1980; Yoder et al. 1983; Alexander 1983; Gasperini et al. 1986) and perhaps more particularly by the publications of the last two or three years (e.g. Peltier & Jiang 1996; James & Ivins 1997; Vermeersen et al. 1997; Mitrovica & Milne 1998), which have dealt with different aspects of the global response: observations of the non-tidal acceleration of the Earth's rotation, of the polar wander velocity and of the rate of change of the Stokes coefficients of the Earth's gravitational potential. A number of aspects of the analysis of the global response to the changing ice loads remain to be examined in detail and this is what this paper sets out to do: to examine the sensitivity of the global response that can be inferred from either satellite motion or planetary rotation to the key parameters that define both mantle rheology and ice load history.

The Earth's gravitational potential is conveniently expressed as a series of spherical harmonic functions whose coefficients, the Stokes coefficients \( C_{20} \), represent integral functions of the mass distribution bounded by the deformable surface including the contributions from the surface load itself, in this case the ice sheets and the ocean waters. Of these coefficients, the principal one is the second-degree, zero-order, longitude-independent (i.e. zonal) coefficient \( C_{20} \) which defines the planet's oblateness. Its primary consequence on the motion of a satellite about the Earth is to induce a rotation of the orbital plane about the Earth's rotation axis, and observations of this motion determine the second-degree zonal Stokes coefficient.

Higher even-degree coefficients contribute to the rotation of the orbital plane as well, but the principal contribution is from \( C_{20} \). The zonal coefficients describe the latitudinal variation of mass within the planet and the longitude dependence is described by the non-zero-order coefficients. In the present context two particularly important coefficients are those of degree 2 and order 1 because these, together with \( C_{20} \), relate to the products and moments of inertia of the planet and, through Euler's equations for a rotating body, determine the rotational responses of the planet to external torques or internal redistributions of mass.

Any modification in the long-wavelength distribution of mass on and within the Earth that contributes to these low-degree Stokes coefficients will therefore have several consequences. A time dependence in the dynamical flattening, \( C_{20} \), results in a secular acceleration in the orientation of the plane of any satellite orbiting the planet, as well as, through the Euler equations, in a modification of the Earth's rotational velocity. A time dependence in the other second-degree coefficients, or equivalently in the products of inertia, results in a change in the direction of the rotation axis relative to the crust.

The orbital perturbation response to the ongoing effects of the last deglaciation has been observed as an acceleration in the orientation of the orbit plane and analysed with the objective of estimating the mantle viscosity, the first result by Yoder et al. (1983) being quickly followed by other analyses (e.g. Alexander 1983; Peltier 1983; Rubincam 1984). The orbital plane intersects the equatorial plane and the line of intersection of the two planes is seen to be rotating at an accelerating rate, superimposed upon which is a rich spectrum of oscillations produced by the other Stokes coefficients in the planet's gravity field. The success of the analysis for the acceleration term therefore rests on the ability to remove this background spectrum, which may contain tidal terms with periods as long as 19 years. The second part of a successful analysis rests on the ability to separate the contributions to the acceleration arising from the other even-zonal harmonics, which may also be undergoing change as a result of the glacial rebound \( (C_{40}, C_{60}, \ldots) \), although it is always \( C_{20} \) that dominates. The first results were based on the observations of a single satellite, LAGEOS, for which the non-glacial rebound perturbations are particularly well known and for which a relatively long observational record exists. These accelerations provide a constraint on a linear combination ('lumped sum') of even zonal harmonics, of which the second-degree term is the dominant one. Subsequent analyses have included other satellites, particularly Starlette, and preliminary attempts have been made to separate out the contributions of \( C_{20}, C_{40} \) and \( C_{60} \) from the observed accelerations (Cheng et al. 1989; Cazenave et al. 1996; Nerem & Klosko 1996).

The Earth's rotation has not been uniform through time and experience, inter alia, a secular change producing a systematic increase in the length of day, superimposed upon which are large-amplitude decadal and longer oscillations. Dicke (1966) suggested that the non-tidal secular acceleration may be a result of the ongoing effects of deglaciation, and the first quantitative analysis was by O'Connell (1971). This acceleration, or, equivalently, a linear change in the length of day, is controlled by the value of \( C_{20} \), and the information on mantle viscosity contained in this observation is very similar to that contained in the satellite observations. One difference is that the latter also contains the contributions from the higher-degree terms, whereas the rotation signal is proportional to \( C_{20} \) only. Hence in principle it becomes possible to separate out the contributions from \( C_{20} \) and the higher-degree terms by combining these two data types. What prevents this from being achieved, and what makes the rotational information of lesser importance for this purpose, is that the secular acceleration in the planet's rotation is influenced by processes other than, and possibly less well quantified than, glacial rebound (e.g. tidal accelerations or core-mantle coupling), and the separation of the contributions is not well constrained (e.g. Lambeck 1980).

Any present-day changes in the ice–ocean balance also contribute to the rotational changes, and Munk & MacDonald (1960, p. 233) calculated the change in the length of day that would be produced by a rise in sea level of 1 cm resulting from melting of Greenland or Antarctic ice. Converting this value to a rate for a sea level rise of 1 mm yr\(^{-1}\) (consistent with tide-gauge observations) yields a value at least as large as the observed secular rate of change. However, because they were considering decade-scale changes in the length of day that are an order of magnitude larger than the secular rate, the present melting of ice sheets was correctly inferred to be too small to...
produce these latter changes. However, if the rising sea level is a consequence of a secular melting of polar ice, this is not correct, nor is it correct to neglect this contribution when interpreting the satellite information on $C_{20}$. Gasperini et al. (1986) included the effects of both present melting and post-glacial rebound and also recognized that if melting has been occurring for hundreds of years, the contribution from present melting could be significantly reduced.

The other observational evidence for the glacial rebound signal is a slow shift in the present position of the rotation axis along a meridional plane of about 70 W at a rate of about $10^{-6}$ °yr$^{-1}$, the ‘polar wander’. Thus the pole in recent geological time was located further away from North America than it is today. The record length is about 100 years and superimposed upon it are decade-length oscillations which, together with changing observational practice during the past century, render some uncertainty to this observation. Numerous geophysical causes for the observed drift have been proposed (e.g. Munk & MacDonald 1960; Lambeck 1980), and its association with an exchange of mass between ice sheets and the oceans was already noted by Munk & MacDonald (1960), although in this case the emphasis was on the contributions from a possible present-day exchange. Dickman (1979) recognized that the Late Pleistocene deglaciation may have contributed to the observed polar wander, and the first quantitative analysis of the problem was by Nakiboglu & Lambeck (1980). Other analyses followed with increasing levels of sophistication (Sabadini & Peltier 1981; Sabadini et al. 1982; Peltier 1982; Wu & Peltier 1984; Yuen et al. 1986) and the formulation of the problem is now well understood. As for the rotation rate information, the interpretation of the observation in terms of mantle viscosity rests on the assumption that other physical processes contributing to changes in the inertia tensor or exerting torques on the planet are either known or small. Whether this is indeed the case can be partly examined by comparing the inferences drawn from the three observation types considered so far or by quantifying some of the other processes.

In discussing the theory for the effects of the past deglaciation on the Earth’s rotation and on satellite motion, a convenient departure point remains Munk & MacDonald (1960), who examined the effects of a present-day exchange of mass between the polar ice caps and the ocean and who provided a simple formulation for the surface loading of a Maxwell viscoelastic sphere of uniform properties (see also Nakiboglu & Lambeck 1980). However, with these models it is only possible to obtain an instantaneous and a secular response, and more complex, multilayer models that include transient behaviour were developed by Sabadini et al. (1982) and Peltier (1982). At that time there was some debate over the validity of certain approximations employed in calculating the polar wander for multilayer models, and, with hindsight, it appears that some of the disagreement stemmed from computational errors, because recent work (Vermeersen & Sabadini 1996; Mitrovica & Milne 1998) demonstrates that it is equally valid to remove the periodic Chandler wobble term from the outset of the derivation or at the end of the calculations. It has long been recognized that the predicted polar wander speed depends on both the lower-mantle viscosity and lithospheric thickness. However, some authors assumed (unreasonably) that the viscosity was very well constrained by observations of postglacial sea level change, and the polar wander data were used to constrain the global average thickness of the lithosphere (Yuen et al. 1983; Peltier 1982). Compressibility of the Earth has been included in only a few of the papers from the earlier period of activity (e.g. Wu & Peltier 1984). Neglecting compressibility produces errors of 10–20 per cent if the lower-mantle viscosity is greater than $3 \times 10^{21}$ Pa s or as much as 50 per cent for a lower-mantle viscosity of $10^{25}$ Pa s (Mitrovica & Milne 1998).

There have been several developments in the modelling of the load history used to calculate polar wander. Sabadini & Peltier (1981) recognized the need to include prior glaciation phases rather than assuming a state of regional isostatic equilibrium at the Last Glacial Maximum. The geometry of the ice load for the earlier work was generally assumed to be circular and the meltwater was uniformly distributed over the oceans. Recent studies of both $C_{20}$ and polar wander have applied more sophisticated load histories (e.g. James & Ivins 1997), revised old predictions based on new ice models, included gravitationally self-consistent ocean loads (Mitrovica & Peltier 1993), increased the detail of Quaternary eustatic sea level variations in the ice model (Peltier & Jiang 1996), etc.

In this paper, we review the theory required to predict $C_{20}$ and polar wander velocity, assess the various approximations in the modelling of the load history and determine how they affect the results. We gain a fundamental understanding of the dependence of the geophysical predictions on the geometric and temporal properties of the load and the Earth rheology by systematically determining how details of the load and earth model contribute to predictions. First, we determine which geometric properties of the load (ice and ocean) are important for the prediction of $C_{20}$ and polar wander. Second, we analyse which temporal parameters of the load and which Earth rheological parameters play significant roles for the predictions. Finally, using independent constraints on the load history and geometry, we predict the contributions to the geophysical observables and their uncertainties as a function of the key rheological parameter (lower-mantle viscosity) and present-day source and rate of melting (for which the observational uncertainties are too large to be able to assume a particular value). This approach confirms the results of previous publications which address the relative importance of rheological parameters but we examine the dependence on load geometry and history more systematically than has been done previously. Therefore, our inference of mantle viscosity is more robust than previous inferences because it encompasses all feasible scenarios of Late Pleistocene and present mass balance changes. The above approach is in contrast to recent studies by Trupin (1993) and James & Ivins (1997), who estimated the uncertainty in the predictions by comparing the results for several different proposed present mass balance scenarios for Antarctica.

## 2 Theory

### 2.1 Changes in the long wavelength component of the geoid

Late Pleistocene deglaciation causes changes in the geoid through the direct effect of redistribution of ice and water on the Earth’s surface and through the redistribution of mass within the Earth caused by glacial isostatic adjustment in response to the changing surface load. Ongoing change in the
long-wavelength component of the geoid can be measured by analysis of the motion of near-Earth satellites such as LAGEOS (Eanes & Bettadpur 1996; Nerem & Klosko 1996; Cazenave et al. 1996). A satellite's orbit is affected by many forces, including the Earth's gravity, solar and lunar attraction and corresponding tides, the solar wind and atmospheric drag. After removing all the known forces, the residual secular acceleration of the node of the satellite orbit is assumed to be caused by a secular rate of change of the zonal harmonics of the Earth's gravitational field (other harmonics do not affect the secular rate of change of the orbit's node). The relative contributions of each of the harmonics depends on the radius, eccentricity and inclination of the orbit, and satellites in low orbits are generally more sensitive to the higher spherical harmonic degrees than those in high orbits.

The fully normalized Stokes coefficients of the geoid ($\tilde{C}_{nm}$) are defined by expanding the gravitational potential $U$ of the Earth into spherical harmonic components (e.g. Heiskanen & Moritz 1967; Lambeck 1988):

$$U(r, \theta, \phi) = -\frac{GmE}{r} \left[ 1 + \sum_{n=2}^{\infty} \left( \frac{a}{r} \right)^n \sum_{m=-n}^{n} \tilde{C}_{nm} Y_{nm}(\theta, \phi) \right], \quad r \geq a,$$

where $(r, \theta, \phi)$ is the position in terms of radius from the centre of the Earth, colatitude and longitude, $a$ and $m_E$ are the radius and mass of the Earth, $G$ is the gravitational constant, $n$ and $m$ are the degree and order of the Stokes coefficient and $Y_{nm}$ is the fully normalized spherical harmonic function. The zonal harmonics $C_{n0}$ of the geopotential are sometimes defined in terms of unnormalized spherical harmonics (e.g. Heiskanen & Moritz 1967) with $J_n = -C_{n0} = -\sqrt{2n+1}C_{n0}$. The rate of change of $C_{nm}$ due to surface loading can be obtained by calculating the rate of change of the gravitational potential at the Earth's surface (e.g. Lambeck 1988):

$$\dot{\tilde{C}}_{nm} = \frac{4\pi a^2 \rho_L}{m_E(2a+1)} \frac{d}{dt} \left[ (1+k_n^0)^0 \Delta L_{nm} \right],$$

where $k_n$ is the load Love number for gravitational potential, $\rho_L$ is the density of the load and $\Delta L_{nm}$ is the fully normalized spherical harmonic component of the load thickness. The asterisk denotes convolution of the time-dependent Love number with the time-dependent load.

Observations of the orbits of satellites do not directly constrain the individual spherical harmonic components of the rate of change of the geoid, but rather a linear combination of them. However, individual low-order zonal components can be obtained by observing several satellites with different orbital parameters because the contribution from higher-degree harmonics decreases rapidly with increasing spherical harmonic degree. In the modelling of glacial rebound, the acceleration of the node of the orbit of the satellite can be predicted by calculation of the components $\dot{\tilde{C}}_{o0}$, including terms to as high a degree as is necessary to obtain convergence, and the predicted secular acceleration of each satellite can be compared directly with the observed value. Nevertheless, for the LAGEOS and Starlette satellites, the acceleration of the node is dominated by the rate of change of the degree 2 zonal Stokes coefficient. For the current precision of observations of $\dot{\tilde{C}}_{o0}$, the same mantle viscosity is inferred whether one tries to fit the observed acceleration of the node of each satellite (i.e. use all harmonics) or only $\dot{\tilde{C}}_{20}$ is fitted (Lambeck & Johnston 1998).

That is, the higher-degree harmonics are not sufficiently well constrained at present to provide a tighter constraint on mantle viscosity than is obtained from the degree 2 harmonic. In particular, different analyses disagree on the sign of $\dot{\tilde{C}}_{30}$ (see Cheng et al. 1997 for a recent summary of observations).

### 2.2 Changes in length of day and polar wander

By changing the mass distribution on and within the Earth, the moment of inertia of the earth–ice–water system is changed. In order to conserve angular momentum, the rotation velocity and the position of the rotation pole change.

The Earth rotates with an almost constant angular velocity $\omega_0$ about the mean rotation pole. Small departures from constant rotation velocity occur, and the rotation vector $\omega$ can be written in terms of dimensionless quantities $m_3$ and the diurnal rotation

$$\omega = \omega_0 (m_1, m_2, 1 + m_3).$$

The rate of change of angular velocity $m_3$ about the mean rotation axis is proportional to the rate of change of the moment of inertia about the rotation axis and is given by

$$m_3 = \frac{I_{33}}{C},$$

where $I$ is the moment of inertia tensor and $A$ (used below) and $C$ are the equatorial and polar moments of inertia. The components of the change in moment of inertia can be written in terms of the spherical harmonic components of degree 0 and 2 of the load. Because mass is conserved in the glacial rebound problem, the degree 0 component of the total ice and water load is zero. The three components of the inertia tensor which are significantly affected by changes in the surface load are (e.g. Wu & Peltier 1984):

$$\Delta I_{33} = -\frac{4\pi a^2 \rho_L}{\sqrt{15}} \left(1+k_3^0\right) \Delta L_{21},$$

$$\Delta I_{23} = -\frac{4\pi a^2 \rho_L}{\sqrt{15}} \left(1+k_3^0\right) \Delta L_{2-1},$$

$$\Delta I_{31} = -\frac{8\pi a^2 \rho_L}{3\sqrt{5}} \left(1+k_3^0\right) \Delta L_{20},$$

where the superscript L indicates the component of the change in inertia caused directly by loading.

In the absence of torques operating at the surface or at the core–mantle boundary, eqs (2), (4) and (7) combine to show that the change in angular velocity is proportional to the rate of change of the degree 2 zonal Stokes coefficient (e.g. Wu & Peltier 1984):

$$m_3 = \frac{8\pi a^2 \rho_L}{3\sqrt{5}C} \frac{d}{dt} \left[ (1+k_3^0) \Delta L_{20} \right] = \frac{2m_I a^2 \sqrt{5}}{3C} \frac{d}{dt} \dot{\tilde{C}}_{20}.$$

$m_3$ is often defined in terms of the rate of change of the length of day $d(\text{lod})/dt$, which is

$$d(\text{lod})/dt = \frac{2\pi}{\omega_0} \frac{d}{dt} \left( \frac{1}{1+m_3} - 1 \right) \equiv -\frac{2\pi}{\omega_0} m_3.$$

Records of ancient and modern eclipses (Stephenson & Morrison 1995) give estimates of the rate of change of the
length of day over the last 2700 years while astrometric measurements give more accurate estimates for the last century (Gross 1996; McCarthy & Luzum 1996).

The two components in eq. (3), \( m_1 \) and \( m_2 \), describe the displacement of the rotation axis in the directions 0° and 90° E longitude respectively and can be combined to form a complex polar motion vector \( \mathbf{m} = m_1 + im_2 \). In a similar fashion, the (2, 1) and (2, -1) spherical harmonic components of the change in load form a complex load vector \( \Delta \mathbf{L} = \Delta L_{21} + i\Delta L_{-21} \) and the off-diagonal components of the inertia tensor form the complex inertia vector \( \Delta I = \Delta I_1 + i\Delta I_2 \). A change in the complex inertia vector is caused by the change in surface load, the yielding of the Earth beneath the load and the change in rotational potential induced by the shift in the rotation axis, \( \Delta I = \Delta I^i + \Delta I^{vol} = \Delta I^i + \frac{(C-A)}{k_f^2} k_f^2 \mathbf{m} \), \( (10) \)

where \( k_f^2 \) is the degree 2 tidal-effective Love number for gravitational potential and \( k_f^2 \) is its value in the fluid limit. The perturbation form of the Euler equation for a planet with a time-dependent inertia tensor is (e.g. Wu & Peltier 1984)

\[
\frac{i}{\sigma_i} \dot{\mathbf{m}} + \mathbf{m} = \frac{\Delta I}{C-A} - \frac{i\Delta I}{\sigma_0(C-A)},
\]

where \( \sigma_i = \frac{C-A}{A} \sigma_0 \)

is the frequency of free nutation of a rigid Earth. For periods much longer than a day, the second term on the right-hand side of eq. (11) can be neglected. Similarly, for periods much longer than the Chandler wobble period (14 months), the first term on the left-hand side can be dropped. Vermeersen & Sabadini (1996) recently clarified a misunderstanding over the removal of this term from the Euler equation (Sabadini et al. 1982; Wu & Peltier 1984). Dropping the first term on the left-hand side removes the oscillatory behaviour of the polar wander (Chandler wobble) that is induced by a non-oscillatory loading history but preserves the mean polar velocity through time. Leaving out the two derivative terms in the Euler equation (11) and substituting from eqs (5), (6) and (10), the polar wander is given by (Munk & MacDonald 1960)

\[
\mathbf{m} = -\frac{4\pi^2 \rho L}{(C-A)\sqrt{15}} \left( 1 - \frac{k_f^2}{k_f^2} \right)^{-1} \left( 1 + k_f^2 \right) \Delta \mathbf{L}.
\]

(12)

### 2.3 Normal-mode representation

For an earth model consisting of uniform incompressible linear viscoelastic layers or shells, the Love number for a delta function load history is a sum of an elastic term and a finite number of decaying exponentials. For example,

\[
k^s_k(t) = k^s_k \delta(t) + \sum_{j=1}^{M} k_{sj}^s \exp(s_{nj}t),
\]

(13)

where \( s_{nj} \) are the inverse relaxation times of the \( M \) modes with amplitude \( k_{nj} \). A similar form can also be used to approximate the behaviour of other laterally homogeneous Earth models with linear rheology (Schapery 1962; Peltier 1974).

To calculate changes in inertia and deformation of the Earth, the load history is convolved with the Earth response, which is described by the Love numbers. By taking the Laplace transform, convolutions simplify to multiplications, so it is simpler to discuss the Earth response in the Laplace domain. The Laplace transform of the equation for the rate of change of the Stokes coefficients (2) is

\[
\bar{\mathbf{c}}_{\text{st}}(s) = \frac{4\pi^2 \rho L}{m_g(2n+1)} \Delta \mathbf{L}_{\text{st}}(s) \left( 1 + k_f^s \sum_{j=1}^{M} \frac{k_f^L}{s-s_{nj}} \right).
\]

(14)

For polar wander, all Love numbers required for the calculation are of degree 2, so the subscript 2 is dropped. The Laplace transform of the equation for polar wander (12) after considerable algebra is (Wu & Peltier 1984)

\[
\bar{\mathbf{m}}(s) = \xi \rho L \Delta \mathbf{L}(s) \left( s + \frac{q_0}{s} + \sum_{j=1}^{M-1} \frac{q_j}{s-s_{nj}} \right).
\]

(15)

with

\[
\xi = -\frac{4\pi^4}{(C-A)\sqrt{15}},
\]

(16)

\[
j_{k} = \frac{k_f^L}{k_f^L - k_f^S} \left( 1 + k_f^S \right),
\]

(17)

\[
q_0 = \sum_{j=1}^{M} \frac{k_f^L}{s_{nf}^2} \sum_{j=1}^{M-1} \frac{k_f^L}{s_{nf(1+2k_f^L)}},
\]

(18)

\[
j_{k} = \frac{k_f^L}{k_f^L - k_f^S} \left( 1 + k_f^S \right) \prod_{j=1, j \neq k}^{M} \frac{\lambda_k - s_j}{\lambda_k - \lambda_j}.
\]

(19)

The \( M-1 \) inverse relaxation times \( \lambda_i \) are the roots of the polynomial

\[
Q(s) = \sum_{j=1}^{M} \frac{k_f^L}{s_{nf}^2} \prod_{k=1, k \neq j}^{M} (s-s_k).
\]

(20)

To obtain expressions for \( \bar{\mathbf{c}}_{\text{st}} \) and polar wander velocity in the time domain, the time dependence of the load must be specified. By interpolating linearly between a discrete set of times \( t_0, t_1, \ldots, t_k \), an arbitrary load history at the surface can be approximated as

\[
\Delta \mathbf{L}(\theta, \phi, t) = \sum_{i=1}^{K} \frac{\Delta \mathbf{L}(\theta, \phi, t_i)}{\Delta t_i} [(t-t_{i-1})H(t-t_{i-1})-(t-t_i)H(t-t_i)],
\]

(21)

where \( H(t) \) is the Heaviside function, \( \Delta \mathbf{L}' = \Delta \mathbf{L} - \Delta \mathbf{L}_{s} \) and \( \Delta \mathbf{L}_{s} = t_i - t_{i-1} \). The Laplace transform of this load history is

\[
\bar{\Delta \mathbf{L}}(\theta, \phi, s) = \sum_{i=1}^{K} \frac{\Delta \mathbf{L}(\theta, \phi, t_i)}{\Delta t_i} \frac{e^{-s_{nj}t_i} - e^{-s_{nj}t_i}}{s^2}.
\]

(22)

By substituting eq. (22) into eqs (14) and (15) and applying the inverse Laplace transform, the rate of change of the Stokes coefficients and of the polar position for the load history \( \Delta \mathbf{L} \).
are given by
\[ \hat{C}_{\text{mm}}(t_k) = \frac{4\pi\alpha^2}{m_0(2n+1)} \left( \frac{\Delta L}{\Delta t} + \sum_{i=1}^{K} \frac{\Delta L_{\text{mm}}}{\Delta t_i} \right), \]
\[ \hat{m}(t_k) = \xi \rho_L \left[ \gamma_i \left( \frac{\Delta L}{\Delta t_k} + \gamma_0 \gamma_i(t_k) + \sum_{i=1}^{K} \frac{\Delta L_{\text{mm}}}{\Delta t_i} \right) \right] \]
where
\[ \gamma_i(s, t_k) = \frac{\exp(s(t_k-t_i))}{s\Delta t_i} - 1. \]
The result for an earlier time can be obtained by including only the load history up until that point in time. For later times, a step with zero change in load can be added to the end of the loading history.

3 RESULTS

The dependence of the predicted values of \( \hat{C}_{\text{mm}} \) and \( \hat{m} \) on the applied load and Earth rheology parameters is discussed in this section. We examine the dependence of predictions first on the location, shape, and profile of the load and second on the time-dependence of the load and Earth rheology. In order to obtain a better understanding of these dependences, we use simple axisymmetric representations rather than realistic load models. Predictions of \( \hat{C}_{\text{mm}} \) and \( \hat{m} \) are shown to depend only weakly on the height profile and radius of axisymmetric loads, but strongly on the total mass and centre of mass of the load. Although realistic ice sheets are not circular in shape, the predictions for an irregular ice sheet lie between those for circular ice sheets which circumscribe and inscribe it. Furthermore, because the predictions depend only weakly on load radius and height profile for circular ice sheets, the predictions for an irregular ice sheet can be adequately approximated using a circular ice sheet of the same mean radius.

The dependence of predictions on the rate and timing of the change in load is discussed in the second part of this section (3.2). We will assume that the load radius, profile and centre remain constant through time and only the height of the ice sheet changes so that the time dependence of the entire load can be described by a scalar function \( \Delta f(t) \) and
\[ \Delta L(t, \phi, t) = L(t, \phi) \Delta f(t). \]
The first two assumptions can be made because the predictions of \( \hat{C}_{\text{mm}} \) and \( \hat{m} \) depend only weakly on radius and height profile. For a single ice sheet, the effect on predictions of a shift in the load centre throughout a glacial cycle is small in comparison with the effect of the change in mass. For the simplified load history (26), the rate of change of the degree 2 zonal Stokes coefficient and rotation pole are given by
\[ \dot{C}_2 = \frac{4\pi\alpha^2}{5m_0} L_{20} R_C, \]
\[ \dot{m} = \xi \rho_L L_{m0} R_m, \]
where the response functions \( R_C \) and \( R_m \) are defined as
\[ R_C = \frac{d}{dt} \left[ (1+k^2)\Delta f \right], \]
\[ R_m = \frac{d}{dt} \left[ \left( \frac{1}{k_f} \right)^{-1} (1+k^2)\Delta f \right]. \]
The geometrical dependence of the predictions are contained in the terms \( L_{20} \) and \( L \) (discussed in Section 3.1), while the dependence on load history and Earth rheology are contained in the response functions \( R_C \) and \( R_m \) (Section 3.2).

3.1 Dependence on load geometry and position

In this section, we examine the dependence of \( L_{20} \) and \( L = L_{21} + iL_{2,-1} \) on parameters of the load geometry \( L(\theta, \phi) \). We make the simplifying assumption that the load is circular with its centre at colatitude and longitude \( (\theta_0, \phi_0) \). The spherical harmonic expansion of the load is then
\[ L(\theta, \phi) = \sum_{n=0}^{\infty} \nu_n Y_{nm}(\theta, \phi), \]
where \( x \) is the angle on the surface of the sphere between \( (\theta, \phi) \) and \( (\theta_0, \phi_0) \) and satisfies
\[ \cos x = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos (\phi - \phi_0) \]
and
\[ Y_{nm}(x) = \frac{1}{\sqrt{2n+1}} \sum_{m=-n}^{n} Y_{nm}(\theta_0, \phi_0) Y_{nm}(\theta, \phi). \]
Therefore, \( L_{20} \) and \( L \) can be written as
\[ L_{20} = \frac{L_2}{\sqrt{5}} Y_{20}(\theta_0, \phi_0) = \frac{L_2}{4} (1 + 3 \cos 2\theta_0), \]
\[ L = \frac{L_2}{\sqrt{5}} \left( Y_{21}(\theta_0, \phi_0) + iY_{2,-1}(\theta_0, \phi_0) \right) = \frac{L_2 \sqrt{3}}{2} \sin 2\theta_0 \cos \phi_0 + i \sin \phi_0, \]
where the dependence on the mass, height profile and radius of the load is contained in the term \( L_2 \) and the dependence on position of the load is contained in the spherical harmonic functions calculated at the load centre.

The mass of an ice sheet \( m_f \) is proportional to the degree 0 spherical harmonic component of the load \( L_0 \):
\[ m_f = 4\pi\rho_L x^2 L_0. \]
It is shown for two different ice profiles below that \( L_2 \) depends mainly on the mass of the ice sheet, but less importantly on how that mass is distributed about the centre of mass of the ice sheet. We consider two different profiles for the ice load, each with maximum radius \( \beta \). Because for realistic ice sheets \( \beta \) is much smaller than \( \pi/2 \) radians, \( L_2 \) can be calculated by a Taylor series expansion.

© 1999 RAS, GJI 136, 537–558
For a parabolic load of maximum thickness $L$ and radius $\beta$, 

$$L(\theta, \phi) = \begin{cases} L \left( \sqrt{1 - \frac{2}{\beta^2}}, \quad \alpha \leq \beta \right), \\ 0, \quad \alpha > \beta \end{cases} \quad \text{and} \quad L_0 = \frac{2}{15} \left( \beta^2 - \frac{4}{63} \beta^4 + O(\beta^5) \right),$$

$$L_2 = \frac{2}{3\sqrt{5}} \left( \beta^2 - \frac{40}{63} \beta^4 + O(\beta^5) \right),$$

and $L_2$ can be written in terms of $L_0$ and the radius $\beta$ as

$$L_2 = L_0 \sqrt{5} \left( 1 - \frac{4}{7} \beta^2 + O(\beta^3) \right).$$

The above height profile corresponds to the change in ice thickness if an ice sheet grows from an initially unglaciated state (e.g. the Laurentide ice sheet). In the second model, we consider a parabolic load of initial thickness $\sqrt{\beta_0/\beta}L$ and radius $\beta_0$, which increases in size to a final thickness $L$ and radius $\beta$ (e.g. the Antarctic ice sheet). The different thicknesses are chosen so that the basal shear stress remains constant (Paterson 1981). If $\beta - \beta_0 \ll \beta$, then

$$L(\theta, \phi) = \begin{cases} L \left( \sqrt{1 - \frac{2}{\beta^2}}, \quad \alpha \leq \beta_0 \right), \\ L \left( \sqrt{1 - \frac{2}{\beta^2}}, \quad \beta_0 < \alpha \leq \beta \right), \\ 0, \quad \alpha > \beta \end{cases} \quad \text{and} \quad L_0 = \frac{1}{3} L (\beta - \beta_0) \left( \beta - \frac{4}{35} \beta^3 - \frac{3}{4} (\beta - \beta_0) \right) + O(\beta^5(\beta - \beta_0), (\beta - \beta_0)^2/\beta),$$

$$L_2 = \frac{\sqrt{3}}{3} L (\beta - \beta_0) \left( \beta - \frac{8}{7} \beta^3 - \frac{3}{4} (\beta - \beta_0) \right) + O(\beta^5(\beta - \beta_0), (\beta - \beta_0)^2/\beta).$$

So, 

$$L_2 = L_0 \sqrt{5} \left[ 1 - \frac{36}{35} \beta^2 + \frac{9}{16} (\beta - \beta_0)^2 + O(\beta(\beta - \beta_0)) \right].$$

In each case $L_2$ is directly proportional to the mass of the load but the ratio $L_2/L_0$ is only weakly dependent on the radius. For example, $L_2$ for a point load is only 6 per cent larger than that for a parabolic load of radius 18° (= 2000 km) of the same mass. This means that for the purposes of predicting $\tilde{C}_{20}$ and $\tilde{n}_0$, the retreat of the ice margins as an ice sheet melts can be neglected provided that the mass change is correctly estimated. The dependence of the ratio $L_2/L_0$ on the profile of the ice sheet is even weaker than the dependence on radius, because the profile only determines the size of the coefficient multiplying the quadratic term in eqs (40) and (44). This is despite the fact that for the first model the largest change in thickness is near the centre of the load, whereas for the second model most melting occurs near the edge of the ice sheet. Therefore, if the change in mass of an ice sheet is known via the eustatic sea level change, an approximate radius and profile of the ice sheet will suffice to predict accurately the load geometry component of $\tilde{C}_{20}$ and $\tilde{n}_0$.

### 3.2 Dependence on load history and Earth rheology

In the previous section, it was shown that predictions of $\tilde{C}_{20}$ and $\tilde{n}_0$ are mainly controlled by the position and mass of an ice sheet. Given that the mass changes much more rapidly than the centre of mass of an ice sheet (that is, the relative change in $m_0$ is much greater than that of $Y_{20}(\theta_0, \phi_0)$ or $(Y_{21} + iY_{2-1}) (\theta_0, \phi_0)$), the approximation (26) of the time dependence of the ice load is reasonable. Therefore, the only parameter of the load remaining which can contribute significantly to the predictions of $\tilde{C}_{20}$ and $\tilde{n}_0$ is the time dependence of the mass of the ice sheet, which is contained in the load history function $\Delta f(t)$. The functions $R_C$ (eq. 29) and $R_m$ (eq. 30) describe the dependence of $\tilde{C}_{20}$ and $\tilde{n}_0$ on the load history and Earth rheology.

Eq. (21) shows how an arbitrary function may be approximated by linearly interpolating between a set of discrete times. The response function for an arbitrary load history function may be obtained by summation of the response to a series of single-step loading histories. So, for simplicity, we examine the response to a single-step loading history, defining $\Delta f$ by

$$\Delta f(t) = \begin{cases} 0, & t < t_0 \\ \frac{t - t_0}{\Delta t}, & t_0 \leq t < t_1 \\ 1, & t_1 \leq t \end{cases} \quad \text{for} \quad t_2 > t_1,$$

where $\Delta t = t_2 - t_1$. For this definition of $\Delta f$, the response functions at a time $t_2$ after the completion of the change in load are given by

$$R_C(t_2) = \frac{d}{dt} \left[ (1 + k^l)^* \Delta f \right] = \sum_{j=1}^{M} k_j \frac{\exp(\gamma_j \Delta t_j) \exp(\gamma_j \Delta t_j - 1)}{\gamma_j \Delta t_j}, \quad t_2 > t_1,$$

$$R_m(t_2) = \frac{d}{dt} \left[ \left( \frac{1 - k^l}{-k^T_j} \right)^{-1} (1 + k^l)^* \Delta f \right] = \frac{k_T}{k_T^j - k^T_j} \left[ \gamma_0 + \sum_{j=1}^{M} \frac{\exp(\gamma_j \Delta t_j) \exp(\gamma_j \Delta t_j - 1)}{\gamma_j \Delta t_j} \right], \quad t_2 > t_1.$$
is chosen because of the strong seismic discontinuity and probable viscosity change due to a change in mineralogy. Some authors have placed boundaries deeper in the mantle (Yuen & Sabadini 1984; Peltier & Jiang 1996; James & Ivins 1997), which leads to slightly different sensitivities of the response functions to mantle viscosity.

It is common to use layered incompressible earth models when calculating $\Delta C_{20}$ and $\mathbf{m}$, but compressibility significantly reduces the polar wander velocity (Vermeersen et al. 1996b; Mitrovica & Milne 1998) and therefore a compressible model is assumed with elasticity and density structure given by the seismological model PREM (Dziewonski & Anderson 1981). The normal-mode representation (14, 15) is approximated using the pure collocation method (Peltier 1974), the details of which are given in Appendix A. At the 670 km discontinuity, we assume a chemical boundary—the effect of a phase change is chosen because of the strong seismic discontinuity and probable viscosity change due to a change in mineralogy. Some authors have placed boundaries deeper in the mantle (Yuen & Sabadini 1984; Peltier & Jiang 1996; James & Ivins 1997), which leads to slightly different sensitivities of the response functions to mantle viscosity.

In order to establish the range of possible values for the response functions, the parameters which are likely to influence the response most strongly are examined first. Of the two load history parameters, the time since the end of loading ($\Delta t_2$) is likely to be more important than the length of loading ($\Delta t_1$), especially a significant time after the loading phase is complete. Of the earth model parameters, the lower-mantle viscosity $\eta_{lm}$ is expected to have the most influence because both $\Delta C_{20}$ and $\mathbf{m}$ occur in response to a spherical harmonic degree 2 deformation which deforms the whole mantle, and the lower mantle has a much larger volume than the upper mantle or lithosphere.

In Fig. 1, the response factors $R_C$ and $R_m$ (in units of kyr$^{-1}$) are contoured as a function of time since the end of loading in the range $0.1 \text{ kyr} \leq \Delta t_2 \leq 100 \text{ kyr}$ and lower-mantle viscosity in the range $10^{21} \text{ Pa s} \leq \eta_{lm} \leq 10^{22} \text{ Pa s}$. All other parameters are held fixed at values of $\Delta t_1 = 0$, $H_I = 80 \text{ km}$, $\eta_{lm} = 3.2 \times 10^{20} \text{ Pa s}$. From the figure, the ranges of predicted response factors are

- $-0.9 \text{ kyr}^{-1} < R_C < 0 \text{ kyr}^{-1}$ and $0 \text{ kyr}^{-1} < R_m < 0.46 \text{ kyr}^{-1}$.

A negative value for $R_C$ means that flattening of the Earth increases after a load is applied near the pole and the positive values of $R_m$ indicate that the pole moves away from a load after it is applied. If $\eta_{lm} = 10^{21} \text{ Pa s}$, the response factors are strongly dependent on the time since loading and are monotonic. For higher values of the viscosity, the response factors are smaller. The strong dependence of the response function on the time since loading implies that small errors in the timing of the deglaciation of a realistic ice sheet may lead to significant errors in the prediction of $\Delta C_{20}$ and $\mathbf{m}$ as shown by e.g. Sabadini & Peltier (1981, Figs. 9 and 8).

For a given length of time since loading, for example $\Delta t_2 = 10 \text{ kyr}$, $|R_C|$ increases as lower-mantle viscosity increases until it reaches a maximum and then decreases again. This behaviour is a consequence of the approximately linear relationship between mantle viscosity ($\eta$) and relaxation time ($1/\eta$). The terms $k_j^t$ are also approximately inversely proportional to mantle viscosity and $R_C$ is a sum of terms of the form $k_j^t \exp (s_j \Delta t_2)$. For a particular value of $\Delta t_2$, each of these terms increases exponentially as viscosity increases until $-1/s_j$ is of the same magnitude as $\Delta t_2$ [that is, $\exp (s_j \Delta t_2) \approx 1$]. Increasing viscosity further leads to a decrease in each term which is inversely proportional to viscosity. The behaviour of $R_C$, being the sum of all such terms, is as shown in Fig. 1 because the dominant relaxation time for deformation of the geoid is less than 10 kyr for a lower-mantle viscosity of $10^{21} \text{ Pa s}$.

The terms $z_j \exp (s_j \Delta t_2)$ forming $R_m$ behave similarly to the components of $R_C$. However, a significant proportion of the rotational relaxation takes longer than 10 kyr for a lower-mantle viscosity of $10^{21} \text{ Pa s}$, and a cross-section through Fig. 1 at $\Delta t_2 = 10 \text{ kyr}$ decreases initially inversely proportionally to viscosity, then flattens out and decreases again. This occurs because there are at least two significant modes of relaxation with distinct relaxation times. The behaviour we

![Figure 1](image-url)
have described for $R_C$ and $R_m$ has been noted by several authors (e.g. O’Connell 1971; Nakiboglu & Lambeck 1980; Yuen et al. 1986).

In Fig. 2, the dependence of the response factors $R_C$ and $R_m$ on lithospheric thickness is examined by contouring them as a function of $\Delta t_2$ for the same range as in Fig. 1 and lithospheric thickness in the range $50 \text{ km} \leq H_l \leq 200 \text{ km}$. The value of $\eta_m$ is held fixed at the same value as before, $3 \times 10^{20}$ Pa s, and there are three panels for values of $\eta_m$ equal to $10^{21}$, $10^{22}$ and $10^{23}$ Pa s. Figs 2(a)–(c) show that $R_C$ is virtually independent of lithospheric thickness except when the lower-mantle viscosity is very high. The dependence on lithospheric thickness at high values of lower-mantle viscosity occurs because flow is restricted to the upper mantle by the high-viscosity lower mantle, and changing the thickness of the lithosphere changes the thickness of the channel in which mantle may flow. $R_m$ is also only weakly dependent on $H_l$ but the dependence is stronger when $\eta_m$ is small. Previous authors (Yuen et al. 1982; Peltier & Wu 1983; Peltier 1984) have drawn attention to an apparent sensitivity of polar wander to lithospheric thickness, and our calculations agree with theirs inasmuch as increasing the lithospheric thickness from 100 to 200 km for a lower-mantle viscosity of $10^{21}$ Pa s and a value of $\Delta t_2$ around 10 kyr increases the polar wander response factor by about 30 per cent. Nevertheless, $R_m$ is much more sensitive to lower-mantle viscosity than lithospheric thickness: an increase of viscosity from $10^{21}$ to $10^{22}$ Pa s decreases $R_m$ to less than half its value.

Fig. 3 shows the dependence of the response factors on upper-mantle viscosity in the range $10^{20}$ Pa s $\leq \eta_m \leq 10^{23}$ Pa s and on time since loading with lithospheric thickness held fixed at $H_l = 80$ km. If $\eta_m = 10^{21}$ Pa s, the response factors are almost independent of $\eta_m$, but the influence of the upper-mantle viscosity on the response factor increases as $\eta_m$ increases because the lower mantle becomes sufficiently rigid for flow to be increasingly restricted to the upper part of the mantle. Nevertheless, for $\Delta t_2 > 5$ kyr, both response factors are rather insensitive to $\eta_m$ for the range of values tested.

Figs 1–3 confirm the results of many previous authors that of the three earth model parameters tested, $\eta_m$ is by far the most important in determining the response factor. This is also the least well-determined parameter from sea level analyses, and the two data types are wholly complementary: observations of $C_{20}$ and $m$ provide constraints on the viscosity of the deep part of the mantle, whereas the sea level observations are generally more sensitive to the shallower parts of the mantle.

The response factors are equally sensitive to the time since loading, $\Delta t_2$. The dependence of the response function on the length of the loading phase $\Delta t_1$ is illustrated for three particular earth models in Fig. 4 as a function of time since the midpoint of the loading phase ($\Delta t_2 + \Delta t_1 / 2$) and the length of the loading phase ($\Delta t_1$). The near-vertical contours in the lower parts of the figures show that if a few thousand years have elapsed since the end of loading, then an instantaneous loading phase has the same response function as a linear loading phase.

**Figure 2.** The response functions $R_C$ (a–c) and $R_m$ (d–f) for a Heaviside load as a function of time since loading ($\Delta t_2$) and lithospheric thickness ($H_l$) for three different values of lower-mantle viscosity.

© 1999 RAS, **GJI** 136, 537–558
Figure 3. The response functions $R_C$ (a–c) and $R_m$ (d–f) for a Heaviside load as a function of time since loading ($\Delta t$) and upper-mantle viscosity ($\eta_{um}$) for three different values of lower-mantle viscosity.

Figure 4. The response function $R_C$ (a–c) and $R_m$ (d–f) for a linear loading phase as a function of time since the midpoint of the load phase ($\Delta t/2$) and the length of the load phase ($\Delta t_1$) for three different values of lower-mantle viscosity.
Therefore, a precise definition of the time dependence of the load in the distant past does not appear to be necessary, although the description of the recent history needs to be more precise to determine the response function accurately for a given earth model.

### 3.2.1 Load cycles

One aspect of the load history which some authors have included in their models (e.g. Peltier 1988; Peltier & Jiang 1996) is the periodic glaciation and deglaciation throughout the Pleistocene. The response factor defined in eqs (29) and (30) can be calculated for several load cycles and the dependence of the response factor on parameters of the loading history, in particular the number of load cycles and the initial load, is explored next.

In Fig. 5, several different candidates are given for the function used to model the load cycle. The functions $f^{l}_{i}$, $f^{u}_{i}$, $f^{h}_{i}$ have sawtooth loading patterns with the loading phase taking $t_{l} = 90$ kyr and the unloading $t_{u} = 10$ kyr and the final unloading finishing at $t_{f} = 7$ ka BP. The superscripts l, u and h stand for the initial load state of loaded, unloaded and half-loaded, respectively, and the subscript i is the number of unloading phases in the load history.

The response functions for each of the load models as a function of the number of load cycles and for three different values of the lower-mantle viscosity are shown in Fig. 6 for $H_{l} = 80$ km and $\eta_{um} = 3.2 \times 10^{20}$ Pa s. For the load histories which were initially loaded or unloaded, the response functions vary with the number of load cycles. Because the only difference between $f^{l}_{i}$ and $f^{u}_{i}$ is the initial loading phase, the difference between these results is simply the response for a linear unloading occurring at i times the load cycle length in the past. This is in agreement with Fig. 1, which shows that $R_{c}$ is very small for changes in load much greater than 10 kyr in the past, while $R_{m}$ decays more slowly with time. The function with a half-loaded initial condition converges more rapidly to the value for many cycles because the half-load is an adequate approximation of the sawtooth load when the time elapsed since the change in load is greater than 200 kyr. Therefore, to model the Quaternary load cycles accurately, we need only include the last two load cycles and the average load before that.

We further investigate the effect on the response function of the timing of the end of the unloading phase $t_{f}$ and its length $t_{u}$ (analogous to $\Delta t_{2}$ and $\Delta t_{1}$ of the previous section). Fig. 7 shows that changing the end of the deglaciation period by 1000 years may change the response functions $R_{c}$ and $R_{m}$ by up to 20 per cent. Note that this sensitivity only occurs for intermediate viscosities for $R_{m}$ in agreement with Fig. 1(b), where the contours of $R_{m}$ are near-vertical for intermediate viscosities but closer to horizontal for other values of the viscosity. Changing the duration of the unloading phase has a smaller effect on each response function as is predicted by the weaker dependence

![Figure 5](image-url)

**Figure 5.** A sawtooth load history with three unloading phases and the initial conditions (a) loaded, (b) unloaded and (c) half-loaded. For this standard load history, each loading phase takes 90 kyr, each unloading phase takes 10 kyr and it is 8 kyr since the end of the final unloading phase.

![Figure 6](image-url)

**Figure 6.** The response functions $R_{c}$ (left) and $R_{m}$ (right) at present for the sawtooth load cycle shown in Fig. 5 as a function of the number of unloading phases and the initial condition of the load for three different values of lower-mantle viscosity.
of the response factors on $\Delta t_i$ than $(\Delta t_j + \Delta t_i)/2$ in Fig. 4. In Fig. 7, a half-loaded initial condition and two load cycles are assumed. From these results, it is apparent that a good approximation of the time dependence of the last deglaciation of the Pleistocene ice sheets is required to determine precisely the glacial isostatic adjustment component of $C_{20}$ and $\mathbf{m}$. 

### 3.2.2. Current changes in load

In eqs (46) and (47), the response functions are given for times after the completion of loading. If the load is changing linearly (i.e. $t_0 < t_2 < t_1$) then the response functions are given by

$$R_c(t_2) = \frac{1}{\Delta t_i} \left(1 + k_2 + \sum_{j=1}^{M} \frac{k_j}{s_j} \exp(s_j(t_2-t_0)) - 1\right), \quad \text{(48)}$$

$$R_m(t_2) = \frac{k_2^T}{(k_2^T - k_j^T)\Delta t_i} \times \left(z_0 + z_0(t_2-t_0) + \sum_{j=1}^{M} \frac{z_j}{s_j} \exp(z_j(t_2-t_0)) - 1\right). \quad \text{(49)}$$

Both $R_c$ and $R_m$ are inversely proportional to the time $\Delta t_i$ or, in other words, are directly proportional to the rate of load change. In Fig. 8, $\Delta t_i R_c$ and $\Delta t_i R_m$ are contoured as functions of the time since the beginning of loading $t_2 - t_0$ and lower-mantle viscosity $\eta_m$. As has been previously shown (Gasperini et al. 1986; Wahr et al. 1993), the viscoelastic response of the Earth reduces the effect on $C_{20}$ if the change in load continues to occur for hundreds to thousands of years. The results for $\eta_m = 10^{21}$ Pa s are in agreement with those of Wahr et al. (1993). For polar motion, the sign of the instantaneous response is the same as that of the viscoelastic response, so the effect on polar motion of a linearly changing load increases with time. The effect of increasing viscosity in each case is to increase the time before viscoelasticity influences the response function. Changing the upper-mantle viscosity or lithospheric thickness has much less effect on the response functions for the range of values previously discussed, and the dependence on these rheological parameters is not plotted.

### 3.3 Combining load geometry and history

In this section, the glacial isostatic adjustment contributions to the present values of $C_{20}$ and $\mathbf{m}$ are calculated as a function of lower-mantle viscosity using estimates of the size, distribution and deglaciation history of the Quaternary ice sheets, and incorporating the results from the previous sections. The contribution from present-day melting, which depends both on mantle viscosity and the assumed present melting rate, is also predicted. Finally, mantle viscosity is inferred assuming that postglacial rebound and present melting of ice sheets are the main contributors to the secular rate of change of the degree 2 Stokes coefficient, and this inference is used to predict the glacial rebound and present sea level change components of polar wander. The observed present values of $C_{20}$ and $\mathbf{m}$ and the current rate of eustatic sea level change are required in order to interpret the predictions, so these observations are discussed first.

Length of day variations occur on a variety of timescales and the amplitude of decadal variations is so large that it is very difficult to determine a secular rate of change from records from this century. These variations are thought to be caused by torque applied to the mantle at the core–mantle boundary (e.g. Rochester 1970). Changes in the degree 2 zonal harmonic of the gravity field are not sensitive to external torques and the relatively good agreement between the value of $C_{20}$ determined from the analysis of near-Earth satellite orbits over the last 20 years and changes in the length of day over the last 2500 years support the argument that it is possible to extract the secular signal from the satellite data, or alternatively that the external torques average out over the millennial timescale (Lambeck

![Figure 7](image-url)
Values of $C_{20}$ obtained from satellite laser ranging observations to satellites are presented in Table 1.

Two recent estimates of the polar wander velocity are given in Table 2. These values are derived from combining data from astrometric observations, very long baseline interferometry (VLBI), laser ranging to artificial satellites and the Moon, and analysis of Global Positioning System (GPS) satellite orbits.

Tide gauge measurements indicate a present rate of eustatic sea level rise of $1^\pm 2$ mm yr$^{-1}$, part of which may be due to changes in the Antarctic and Greenland ice sheets as well as mountain glaciers (e.g. Meier 1984) and part of which may be due to thermal expansion near the surface of the oceans (e.g. Church et al. 1991). Some recent estimates of the present mean global rate of sea level rise are given in Table 3.

As shown in Sections 3.1 and 3.2, the load parameters which most strongly influence the predictions of $C_{20}$ and $m$ are the total volume of ice and the timing of the final deglaciation, the longitude and latitude of the centres of each ice sheet and any recent melting. The change in eustatic sea level since the Last Glacial Maximum (LGM) is $125^\pm 5$ m as determined from sea level observations at sites distant from the Late Pleistocene ice sheets (Fleming et al. 1998). These observations constrain the timing of deglaciation to an accuracy of the order of a thousand years or less and the height of the sea level to within $5^\pm 10$ m throughout the late glacial period. Some authors (Nakada & Lambeck 1988; Lambeck 1997) have observed that models which stop melting abruptly 6000 years ago consistently overestimate the relative sea level during the postglacial period, and they suggest that up to 3 m of eustatic sea level rise has occurred over the last 6000 years, probably due to melting in Antarctica. The locations of the last great ice sheets are well constrained from geomorphological observations, but because it is difficult to determine how thick an ice sheet may have been throughout its history, there are uncertainties of the order of 20–30 per cent in the LGM volume of each individual ice sheet. Generally, the timing of deglaciation is better constrained by geomorphological observations of the retreat of each ice sheet. The volume of the Antarctic ice sheet at LGM is the least well constrained with

Table 1. Inferred values of $C_{20}$ from observations of low-orbit satellites.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Satellite</th>
<th>$C_{20}$ ($\times 10^{-19}$ s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yoder et al. (1983)</td>
<td>Lageos I</td>
<td>$5.0 \pm 0.4$</td>
</tr>
<tr>
<td>Rubincam (1984)</td>
<td>Lageos I</td>
<td>$3.7 \pm 0.8$</td>
</tr>
<tr>
<td>Cheng et al. (1989)</td>
<td>Starlette</td>
<td>$3.5 \pm 0.4$</td>
</tr>
<tr>
<td>Gegout &amp; Cazenave (1991)</td>
<td>Lageos I</td>
<td>$3.94 \pm 0.06$</td>
</tr>
<tr>
<td>Cazenave et al. (1996)</td>
<td>Lageos I &amp; II</td>
<td>$4.3 \pm 0.7$</td>
</tr>
<tr>
<td>Nerem &amp; Klosko (1996)</td>
<td>Lageos I &amp; II, Ajsai &amp; Starlette</td>
<td>$3.93 \pm 0.35$</td>
</tr>
<tr>
<td>Eanes &amp; Bettadpur (1996)</td>
<td>Lageos I &amp; Starlette</td>
<td>$3.63 \pm 0.48$</td>
</tr>
<tr>
<td>Cheng et al. (1997)</td>
<td>eight satellites</td>
<td>$3.8 \pm 0.6$</td>
</tr>
</tbody>
</table>

Table 2. Inferred values of $m$ from astronomical and satellite observations.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Polar wander speed (marsec yr$^{-1}$)</th>
<th>Polar wander direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross (1996)</td>
<td>$3.2 \pm 0.2$</td>
<td>$(66 \pm 3)$ W</td>
</tr>
<tr>
<td>McCarthy &amp; Luzum (1996)</td>
<td>$3.33 \pm 0.08$</td>
<td>$(75.0 \pm 1.1)$ W</td>
</tr>
</tbody>
</table>

Table 3. Estimates of the present rate of sea level change from tide gauge observations.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Rate of sea level rise (mm yr$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gornitz et al. (1982)</td>
<td>$1.2 \pm 0.1$</td>
</tr>
<tr>
<td>Barnett (1984)</td>
<td>$2.3 \pm 0.2$</td>
</tr>
<tr>
<td>Peltier &amp; Tushingham (1989)</td>
<td>$2.4 \pm 0.9$</td>
</tr>
<tr>
<td>Emery &amp; Aubrey (1991)</td>
<td>$1.5 \pm 1.5$</td>
</tr>
<tr>
<td>Nakiboglu &amp; Lambeck (1991)</td>
<td>$1.15 \pm 0.38$</td>
</tr>
<tr>
<td>Douglas (1997)</td>
<td>$1.8 \pm 0.1$</td>
</tr>
</tbody>
</table>
estimates based on sea level observations in Antarctica and glaciological models ranging from 8–12 m (Zwartz 1995) and 12–16 m (Huybrechts 1990) to 25–30 m (Hughes et al. 1981).

To estimate the uncertainties associated with the geometrical factors $L_{z0}$ and $|L|$, the values for the change in these parameters between the LGM and the present have been given for two commonly used ice models in Table 4. ARC3 (Nakada & Lambeck 1988) consists of the ICE-1 (Peltier & Andrews 1976) model for North America, Greenland and Fennoscandia with an additional Barents and Kara seas ice sheet similar in magnitude to the Fennoscandian ice sheet. The ANT3 ice sheet was generated from Denton & Hughes’ (1981) maximum reconstruction for Antarctica. The total eustatic sea level change for ARC3 is 127 m. The ICE-3G model (Tushingham & Peltier 1991) is a later generation of the ICE-1 model with additional ice sheets in the Barents and Kara seas, Siberia and Antarctica and with a total eustatic sea level change of 112 m since the LGM. Its LGM sea level is derived by fitting the observed sea level at Barbados but not allowing for the approximately 6 m of tectonic uplift which has occurred at that site since the LGM. In Table 4, Iceland is included as part of the Greenland ice sheet and the British Isles are included as part of the Fennoscandian ice sheet.

The most significant differences between the two ice models are that ICE-3G has much smaller Fennoscandian and Antarctic ice sheets than ARC3 and a larger Greenland ice sheet, and that ICE-3G also has an eastern Siberian ice sheet. The fourth column of Table 4 shows the eustatic sea level rise associated with the melting of each ice sheet. The differences in these individual ice sheets leads to a large difference in eustatic sea level at the LGM of 15 m.

In order to predict $C_{20}$ and $\bar{n}$ and their errors for the Late Pleistocene deglaciation, the ESL contribution of each ice sheet and the covariance between them are estimated by assigning the average ESL contributions of the two ice models ICE-3G and ARC3 to each ice sheet with an $a$ priori uncorrelated errors of 10–100 per cent and estimating the midpoint of melting $t_m$ and length of melting $t_l$ for a linear melting phase. Then an improved $a$ posteriori estimate of the ice sheet parameters (ESL, $t_m$, $t_l$) and covariance matrix is obtained by solving a constrained least squares problem with the constraint that the sum of the ice sheets fits the observed eustatic sea level change (cf. Cross 1983). In this procedure, a set of eustatic sea level constraints was obtained by interpolating from a eustatic sea level curve derived from far-field sea level observations corrected for isostatic vertical movements (Fleming et al. 1998). Fig. 9 shows both the $a$ priori and $a$ posteriori eustatic sea level curves calculated from the ice sheet melting models and the eustatic sea level constraints, while Table 5 shows the $a$ priori and $a$ posteriori ice sheet parameters for each ice sheet. In order to improve the fit to the eustatic sea level constraints in the $a$ posteriori ice model, a small increase in the size of all of the ice sheets, a delay of the melting by up to 1100 years and a continuation of melting until around 4 ka BP are required.

In Section 3.1, it was noted that the ratios $L_{z0}/L_0$ and $L/L_0$ are chiefly functions of the latitude of the centre of the ice sheet and depend only weakly on other factors such as the radius and profile. Estimates of these ratios for each ice sheet are calculated by taking the average of the ARC3 and ICE-3G values from Table 4, and because their uncertainties are much smaller than the uncertainties in ice volumes, the ratios are

![Figure 9. The $a$ priori eustatic sea level curve derived from estimates of ice volumes and the $a$ posteriori curve derived from estimates of ice volumes combined with eustatic sea level constraints.](image)

### Table 4. Low degree spherical harmonic components of the ARC3, ANT3 and ICE-3G ice models in units of metres.

| Region         | Ice model | $L_0$ | ESL  | $L_{z0}$ | $L_{21}$ | $L_{2-1}$ | $|L|$ |
|----------------|-----------|-------|------|----------|----------|-----------|------|
| North America  | ARC3      | 46.2  | 59.4 | 56.2     | −1.5     | −74.5     | 74.5 |
|                | ICE-3G    | 42.6  | 54.9 | 53.6     | −3.2     | −67.4     | 67.5 |
| Greenland      | ARC3      | 2.7   | 3.5  | 5.0      | 2.6      | −19.2     | 3.2  |
|                | ICE-3G    | 5.1   | 6.6  | 9.8      | 4.1      | −3.4      | 5.4  |
| Fennoscandia   | ARC3      | 11.7  | 15.0 | 17.5     | 16.2     | 6.9       | 17.6 |
|                | ICE-3G    | 5.9   | 7.6  | 9.3      | 8.1      | 3.5       | 8.8  |
| Barents and Kara| ARC3     | 9.2   | 11.8 | 17.8     | 3.3      | 8.6       | 9.2  |
|                | ICE-3G    | 9.4   | 12.1 | 18.2     | 4.1      | 8.4       | 9.3  |
| Siberia        | ARC3      | 0.0   | 0.0  | 0.0      | 0.0      | 0.0       | 0.0  |
|                | ICE-3G    | 2.2   | 2.8  | 4.3      | −2.0     | 0.6       | 2.1  |
| Antarctica     | ANT3      | 28.8  | 37.1 | 57.8     | −2.5     | 4.0       | 4.7  |
|                | ICE-3G    | 21.5  | 27.7 | 43.5     | −1.2     | 2.8       | 3.1  |
| Total          | ARC3 + ANT3| 98.6  | 126.8| 154.0    | 18.1     | −56.9     | 59.7 |
|                | ICE-3G    | 86.8  | 111.8| 138.7    | 9.9      | −55.6     | 56.5 |

© 1999 RAS, GJI 136, 537–558
assumed to be known precisely. Using these ratios and the estimates of $L_0$ obtained by fitting the eustatic sea level curve, values of $L_{20}$ and $L$ are estimated.

The contribution of Late Pleistocene deglaciation to $\dot{C}_{20}$ and $\dot{m}$ is now calculated as a function of lower-mantle viscosity using the response functions $R_C$ and $R_m$ calculated in Section 3.2 and the a posteriori model of deglaciation with two earlier glacial cycles of period 100 kyr and an initial state halfway between the glacial maximum and the interglacial. The results, assuming $H_t=80$ km and $\eta_m=3.2 \times 10^{20}$ Pa s, are shown in Fig. 10. The standard errors due to uncertainties in the a posteriori ice model are shown by the dashed lines and are derived from the covariance matrix of the ice model parameters and the derivatives of the response functions $(R_C, R_m)$ with respect to each ice model parameter $(L_0, \eta_m, t)$. For a particular lower-mantle viscosity, the postglacial rebound contributions to $\dot{C}_{20}$ and $|\dot{m}|$ are constrained to within $\pm 4$ per cent and $\pm 12$ per cent, respectively, despite the relatively poor constraints on the melting histories of some of the ice sheets. $\dot{C}_{20}$ is particularly well constrained because the geometry factors for polar wander for those two ice sheets are quite different (melting of Antarctica has little effect on polar wander, whereas polar wander velocity is quite sensitive to melting of the North American ice sheet).

The results in Fig. 10 include only the effects of the ice load. However, the water load contributes approximately 15–20 per cent of the total (Mitrovica & Peltier 1993) and is calculated by using circular ice models with parabolic profiles approximating realistic ice sheets on a real Earth. Their sizes and positions are determined from the average values of the degree 0 and 2 harmonics of ANT3 and ICE-3G. The percentage uncertainties in the predictions that were obtained from the method above are applied to the results obtained including the water load and are given in Fig. 11.

In order to compare predictions with observations, the contributions to $\dot{C}_{20}$ and $|\dot{m}|$ due to melting of the Late Pleistocene ice sheets must be added to the contribution from ongoing melting. The most likely regions for significant present melting are Antarctica, Greenland and mountain glaciers. Because all three sources are at high latitudes, they have a similar geometry factor for $\dot{C}_{20}$, whereas the geometry factor for $|\dot{m}|$ depends more strongly on the longitude of the load, but is quite small because these ice masses are situated close to the rotation axis (see eq. 35). The relatively even distribution of mountain glacier discharge about the rotation axis results in a contribution of at most 10 per cent of the total observed polar wander signal (Lambeck 1980; Gasperini et al. 1986; Trupin et al. 1992) and can be neglected. The contribution to $\dot{C}_{20}$ and $|\dot{m}|$ is also determined by the rate of melting and the length of the melting period (Gasperini et al. 1986; Wahr et al. 1993). There is uncertainty in the present rate of eustatic sea level change because steric changes caused by changes in density of the water do not contribute to either $\dot{C}_{20}$ or $|\dot{m}|$ but do contribute to the observed sea level rise of approximately 1.5 mm yr$^{-1}$. The steric contribution is approximately 0.7 mm yr$^{-1}$ (Church et al. 1991; Nakiboglu & Lambeck 1991). It has also been shown that postglacial rebound contaminates the inferred

---

**Table 5. A priori and a posteriori estimates of ice parameters.**

<table>
<thead>
<tr>
<th>Region</th>
<th>ESL (m)</th>
<th>$t_m$ (ka BP)</th>
<th>$\eta$ (kyr)</th>
<th>ESL (m)</th>
<th>$t_m$ (ka BP)</th>
<th>$\eta$ (kyr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>North America</td>
<td>57±6</td>
<td>11±0.75</td>
<td>8±1.5</td>
<td>60.9±5.0</td>
<td>10.2±0.5</td>
<td>7.4±1.1</td>
</tr>
<tr>
<td>Greenland</td>
<td>5±1.5</td>
<td>10±1.5</td>
<td>8±3</td>
<td>5.3±1.5</td>
<td>9.8±1.4</td>
<td>7.7±2.9</td>
</tr>
<tr>
<td>Fennoscandia</td>
<td>11±1.5</td>
<td>12±0.75</td>
<td>7±1.5</td>
<td>11.2±1.5</td>
<td>12.0±0.7</td>
<td>7.0±1.5</td>
</tr>
<tr>
<td>Barents and Kara</td>
<td>12±5</td>
<td>12±0.75</td>
<td>7±1.5</td>
<td>14.0±4.5</td>
<td>11.7±1.4</td>
<td>7.7±2.8</td>
</tr>
<tr>
<td>Siberia</td>
<td>1.5±1.5</td>
<td>12±1.5</td>
<td>7±3</td>
<td>1.7±1.5</td>
<td>12.0±1.5</td>
<td>7.0±3.0</td>
</tr>
<tr>
<td>Antarctica</td>
<td>20±6</td>
<td>11±1.5</td>
<td>10±3</td>
<td>24.2±5.0</td>
<td>9.9±0.9</td>
<td>11.5±2.1</td>
</tr>
</tbody>
</table>

---

**Figure 10.** Inferences of $\dot{C}_{20}$ and $|\dot{m}|$ due to the a posteriori deglaciation model as a function of lower-mantle viscosity (solid line) with estimated uncertainty (dashed lines) derived from the a posteriori covariance matrix of the ice volumes.
eustatic sea level (Lambeck & Nakiboglu 1984; Peltier & Tushingham 1991), so there is considerable uncertainty in
the present rate of non-steric sea level change.

Focusing first on the predictions of $\tilde{C}_{20}$, and assuming that
postglacial rebound and recent melting are the causes of the
geophysical signal, the lower-mantle viscosity can be inferred
given the present rate of sea level rise, the duration of recent
melting and assuming a geometry factor of $L_{20}/L_0 \approx 2.38$. The
geometry factor includes the effect of the water load which
adds approximately 18 per cent to the geometry factor from
melting of ice from either Antarctica or Greenland. Fig. 12
plots the range of lower-mantle viscosity values which fit the
observed $\tilde{C}_{20}$ of $(3.9 \pm 0.4) \times 10^{-19}$ s$^{-1}$ for different values of
the present melting rate. The results in Fig. 12 show that there
are two solutions if the present rate of sea level rise is less
than 1.4 mm yr$^{-1}$ and no solution if the rate is greater than
1.7 mm yr$^{-1}$ (in agreement with many previous authors,
particularly the qualitative Fig. 10 of Sabadini et al. 1988). The
branch of the solution with higher viscosity is unlikely to be the
true solution as it conflicts with most inferences of mantle
viscosity from sea level data, although the sea level data are
primarily sensitive to the shallow part of the lower mantle.
On the lower branch, the inferred viscosity increases with
increasing present rate of sea level change (Sabadini et al. 1988)
and decreases the longer the present phase of melting has been
occurring (Gasperini et al. 1986; Wahr et al. 1993). Assuming
that no other process contributes substantially to the observed
$\tilde{C}_{20}$, the present rate of sea level change can be inferred to be
less than 1.6 mm yr$^{-1}$, in agreement with Mitrovica & Peltier
(1993). The behaviour of the solution is the same as that found
by James & Ivins (1997) but because we do not include the
steric sea level change as part of the total sea level change,
0.4 mm yr$^{-1}$ (their assumed steric sea level change rate)
should be subtracted from their sea level change estimates to
make a direct comparison with our results. They also include
0.38 mm yr$^{-1}$ sea level rise from the melting of mountain
glaciers. Because the mountain glaciers are situated at a
lower latitude, they have a smaller geometry factor with a

Figure 11. Inferences of $\tilde{C}_{20}$ and $\dot{\mathbf{m}}$ due to the afteriori
deglaciation model including ocean loading as a function of lower-mantle viscosity
(solid line) with uncertainties (dashed lines) derived from Fig. 10.

Figure 12. Lower-mantle viscosity inferred by fitting the predicted value of $\tilde{C}_{20}$ to the observed value as a function of the present rate of sea level rise
and the duration of recent loading. Results are given for multiples of 0.1 mm yr$^{-1}$ sea level rise but symbols are offset slightly for presentation
purposes.
correspondingly smaller contribution to the $\dot{C}_{20}$ signal and therefore James & Ivins (1997) obtain a slightly higher estimate of the maximum present rate of melting. The only solutions for which the present rate of sea level is greater than 1.6 mm yr$^{-1}$ requires a duration of melting of at least 2000 years, which gives a total change of 3.2 m over that period, which is inconsistent with the geological record. Fig. 12 shows that a tighter constraint on the present rate of sea level change will enable us to infer the mantle viscosity much more precisely. If the present rate of non-steric sea level rise is 1.0 mm yr$^{-1}$ and the duration of melting is less than 1000 years, then the mantle viscosity is estimated as log$_{10} \eta_{\text{mantle}} = 21.82 \pm 0.15$. Estimates can similarly be obtained for other present melting rates, but if the current rate of sea level rise is small, a longer melting period can be allowed without conflicting with the observation of less than a metre of sea level change over the last 4000 years. Note that the observed sea level fall at Stockholm over the last century (1885–1984) is 1 mm yr$^{-1}$ less than that for the previous century (1774–1884) (Ekman 1988), which suggests that the observed eustatic sea level rise of 1–2 mm yr$^{-1}$ is only representative of the rate over the last century.

The final column of Table 4 (IL), which is proportional to polar wander velocity, shows the effectiveness of each Pleistocene ice sheet in exciting polar wander. Despite its much smaller change in mass, the Greenland ice sheet contributes more to polar wander than the Antarctic ice sheet because Greenland is further from the pole. Therefore, we might expect that present-day melting of Greenland would contribute more strongly to the polar wander signal than melting of Antarctica. The current mass balance of Antarctica is not well constrained; the accumulation due to precipitation is equivalent to approximately 7 mm yr$^{-1}$ sea level fall (e.g. Trupin 1993) and the mass loss due to iceberg calving and ablation is about the same with an uncertainty of about 10 per cent. Both accumulation and mass loss are largest at the edge of Antarctica. If there are significant differences between the centres of mass of accumulation and ablation as in the models proposed by Trupin (1993), then the mass balance of the Antarctic ice sheet can easily produce polar wander comparable in magnitude with the observed signal. This is further demonstrated by James & Ivins (1997), who calculated the contributions to $\dot{C}_{20}$ and $\dot{C}_{22}$ for a range of current Antarctic mass-balance scenarios. Their preferred Antarctic melting model J92 has a similar geometry factor to their melting scenario for NE Greenland; their scenario 1 has an even larger geometry factor, while scenarios 2 by mass and area both have small geometry factors. A large geometry factor is obtained if the centre of mass change is a large distance (≈ 10$^3$ km) from the pole. Both scenario 1 and J92 are dominated by large mass changes in West Antarctica and little change in East Antarctica, while both of the scenarios 2 have a more even distribution of mass change. If the Antarctic geometry factor is large and similar in direction to the Greenland geometry factor, then polar wander, like $\dot{C}_{20}$, will not help to locate the source of present melting because the geophysical signals from Greenland and Antarctic melting would be indistinguishable. A precise determination of $\dot{C}_{20}$ would be much more effective at locating the melt source (James & Ivins 1997). Nevertheless, it is likely that the centres of mass of accumulation and ablation do coincide, in which case we need only consider changes in mass equivalent to approximately 1 mm yr$^{-1}$ sea level change and with a well-determined geometry factor. In the following calculations we will assume that the centres of mass of the present deglaciation for Greenland and Antarctica coincide with the centres of mass of the change since the LGM.

The combination of mantle viscosity and present rate of sea level change obtained from the $\dot{C}_{20}$ observation can be used to infer how much polar wander is predicted to occur at present due to Late Pleistocene deglaciation and present melting. Because the polar wander geometry factors for melting from Greenland and Antarctica differ significantly, the proportion of the present melting coming from Greenland plays an important role in determining the predicted present polar wander velocity and direction (subject to the assumption that the centres of ablation and accumulation in Antarctica coincide). In Fig. 13, the predicted present polar wander speed and direction are contoured as functions of lower-mantle viscosity and the fraction of melt coming from Greenland. Three different rates of present melting encompassing the range inferred from tide gauge observations are considered. The lower branch of the viscosity range inferred from fitting the predicted $\dot{C}_{20}$ to the observed value is shaded. Increasing the present rate of melting increases the predicted polar wander velocity. Melting from Antarctica has little effect on the direction, whereas melting from Greenland shifts the polar wander direction to the east of 70°. The results in Fig. 13 are for the case where the present melting has lasted for only 0.1 kyr. However, the results depend only weakly on the period of recent melting.

Although it has been shown that processes such as subduction (Spada et al. 1992), mountain building (Vermeersen et al. 1994) and mantle convection (Steinberger & O’Connell 1997) may contribute significantly to the observed polar wander, it is tempting to explain the entire signal in terms of postglacial rebound and present melting of ice sheets. If this is a valid assumption and the estimated geometry factor for Antarctica is correct, then we can estimate the source and rate of melting and the average mantle viscosity from the observations of $\dot{C}_{20}$ and the direction ($\sim 70^\circ$) and velocity $(3.2 \pm 0.2$ massecyr$^{-1}$) of polar wander. In all cases, the present direction of polar wander indicates that most of the melting comes from Antarctica—significant melting from Greenland shifts the polar wander direction too far to the east. The polar wander velocity is fitted best if the present rate of sea level change is at the high end (i.e. $\sim 1.5$ mm yr$^{-1}$). For lower rates of sea level change the polar wander velocity is too fast. If this inference is correct, the lower-mantle viscosity is also inferred to be quite high: $(1-3) \times 10^{22}$ Pa s. The inference of the rate of sea level change relies heavily on the assumption that the geometry factor for Antarctica is much smaller than that for Greenland. If the geometry factor for Antarctica has been underestimated by a factor of 2, for example, then the inferred rates of sea level are overestimated by the same factor. However, the conclusion that most of the melting must come from Antarctica still stands because current melting in Antarctica does not change the direction of polar wander from the direction due to Late Pleistocene deglaciation whereas Greenland melting does.

4 CONCLUSIONS

By examining predictions of $\dot{C}_{20}$ and polar wander speed and direction in response to relatively simple load models, the importance of various parameters of the load and earth models...
has been investigated. The mass and latitude of the centre of mass are the important geometrical factors, with the longitude of the centre of mass determining the direction of polar wander. The predictions are relatively insensitive to the distribution of ice about the centre of mass. The length of time since the midpoint of the deglaciation period is extremely important, but the length of the unloading period is less so. Also, many glacial cycles may be adequately approximated by the mean load for the period prior to the last interglacial without significantly modifying predictions. The average lower-mantle viscosity is the main rheological parameter which determines \( C_{20} \) and polar wander velocity due to Late Pleistocene deglaciation. Polar wander is slightly sensitive to lithospheric thickness, whereas \( C_{20} \) is insensitive to this parameter. Predictions of both \( C_{20} \) and polar wander are also insensitive to the value of upper-mantle viscosity within plausible limits, and inferences of lower-mantle viscosity using the observed value of \( C_{20} \) are independent of the assumed values of both upper-mantle viscosity and lithospheric thickness. Present melting of ice sheets and mountain glaciers also contributes strongly to \( C_{20} \) predictions and may contribute strongly to polar wander velocity if the source of melting is at a sufficiently low latitude.

Because the response factor for predictions of \( C_{20} \) decays to small values for very long timescales, it is reasonable to assume that most of the present signal is due to current melting and the Late Pleistocene deglaciation. Allowing for uncertainties in the masses of the Late Pleistocene ice sheets and the timing of deglaciation, we have inferred the average lower-mantle viscosity as a function of the present rate of eustatic sea level rise due to melting of high-latitude ice sheets. As other authors have found, there are two solutions, one of which has a very high viscosity and can be discounted as incompatible with the evidence from relative sea level observations. The inferred mantle viscosity correlates with the assumed present rate of sea level rise and length of the unloading period and can be read from Fig. 12. We have also considered in greater detail than previous authors the effect of the period of recent loading. Increasing the period of unloading reduces the inference of mantle viscosity on the lower branch and increases it slightly on the upper branch. If the present rate of sea level rise is 1 mm yr\(^{-1}\) and the duration of the recent melting is less than 1000 years, then we infer a lower-mantle viscosity satisfying \( \log_{10} \eta_{lm} = 21.82 \pm 0.15 \), a result that is consistent with inferences drawn from sea level analyses (e.g. Nakada \& Lambeck 1988; Lambeck et al. 1996, 1998). The contribution to polar wander for viscosity-melting rate pairs which fit the observation of \( C_{20} \) give a polar wander velocity close to the observed value only if most of the melting is from Antarctica and if the current rate of non-steric sea level change is close to 1.5 mm yr\(^{-1}\). Although other geophysical processes are capable of causing polar wander, this result strongly suggests that Late Pleistocene deglaciation and current sea level changes are the dominant contributors to present-day polar wander.

In Appendix B, we show that the presence of internal phase boundaries in the Earth has negligible effect on predictions of \( C_{20} \) and reduces predictions of the glacial rebound component of polar wander by up to 30 per cent.

---

**Figure 13.** The predicted polar wander velocity and direction due to Late Pleistocene and present melting as a function of lower-mantle viscosity and the fraction of the present melt coming from Greenland for three different present melting rates. The shaded regions represent the inferred viscosity range for each melting rate.
REFERENCES


APPENDIX A: AN APPROXIMATE METHOD TO CALCULATE LOVE NUMBERS AND POLAR WANDER—THE PURE COLLOCAION METHOD

An approximate method to calculate the Love numbers for an earth model which has continuously varying material properties is used because there are no analytical solutions and because the normal-mode method is difficult to employ as many layers are required to approximate the real Earth to sufficient accuracy. This requires finding many (hundreds of) zeroes of the secular determinant, which is troublesome for most root finders (Vermeersen et al. 1996a). The pure collocation method (Schapery 1962; Peltier 1974) circumvents this problem by choosing a range of relaxation times a priori and calculating the relative strengths of each relaxation time which best approximate the Love numbers in the Laplace transform domain. The form of the pure collocation method solution is exactly the same as the analytic solution for a layered incompressible earth model and is exact when the assumed relaxation times coincide with those for the analytic solution. The advantage of the pure collocation method is that a very good approximate solution can be obtained with fewer relaxation times.
In the pure collocation method, a set of inverse relaxation times \( \{ \sigma_1, \sigma_2, \ldots, \sigma_N \} \) are chosen along the positive real axis, usually at equal spacing on a logarithmic scale at two or three points per decade (e.g. Mitrovica & Peltier 1992). Then corresponding weights are calculated so that the Laplace transform of the quantity being calculated is equal to the approximate solution at the values \( \sigma_i \). The form assumed for the Love numbers is

\[
\tilde{f}(s) = f_0 + \sum_{j=1}^{N} \frac{f_j}{s + \sigma_j}.
\]  

(A1)

The weights \( f_j \) can be calculated by substituting \( N \) different values of \( s \) into eq. (A1). The obvious choice of the \( N \) values of \( s \) are the values \( \{ \sigma_1, \sigma_2, \ldots, \sigma_N \} \). Defining the matrix \( a_{ij} = f_j/(\sigma_i + \sigma_j) \) and the vectors \( r_j = f_j/\sigma_j \) and \( \mathbf{b}_i = f(\sigma_i) - f_0 \), the following system of linear equations is satisfied:

\[
b_i = \sum_{j=1}^{N} a_{ij} r_j, \quad i = 1, 2, \ldots, N.
\]  

(A2)

The definition of the matrix \( (a_{ij}) \) is chosen slightly different from the usual form \( a_{ij} = 1/(\sigma_i + \sigma_j) \) to reduce its condition number and improve stability of the calculation. These definitions can be used directly to calculate the tidal-effective and load Love numbers.

The expression for polar wander includes an additional term \( 20/\pi \) which does not appear in the normal-mode form for the Love numbers (see eqs 14 and 15). In this case, the value \( 20/\pi \) can be calculated from the collocation method approximation and then absorbed into the term on the left-hand side of eq. (A2). The same set of relaxation times \( \{ \sigma_i \} \) can be assumed for the collocation approximation of polar wander. First, we define the weights for the degree 2 potential load and tidal-effective Love numbers:

\[
\tilde{k}^L(\sigma_i) - k^L_0 = \sum_{j=1}^{N} a_{ij} r_j, \quad i = 1, 2, \ldots, N.
\]  

(A3)

\[
\tilde{k}^T(\sigma_i) - k^T_0 = \sum_{j=1}^{N} a_{ij} t_j, \quad i = 1, 2, \ldots, N.
\]  

(A4)

Then,

\[
20/\pi = \sum_{j=1}^{N} t_j/\sigma_j
\]  

(A5)

and

\[
\tilde{m}(\sigma_i) = \frac{1 + \tilde{k}^L(\sigma_i)}{1 - k^L(\sigma_i)/k^L_0} \cdot \frac{k^L_0(1 + k^L_0 + \sum_{j=1}^{N} t_j a_{ij})}{\sum_{j=1}^{N} t_j a_{ij}}, \quad i = 1, 2, \ldots N.
\]  

(A6)

**APPENDIX B: THE EFFECT OF INTERNAL PHASE BOUNDARIES ON \( \dot{C}_{20} \) AND POLAR WANDER**

The buoyancy force arising from the density jumps at the 670 and 420 km depth boundaries within the Earth contribute to both \( \dot{C}_{20} \) and polar wander. It is of particular importance for polar wander because the mode due to the density contrast at the surface, which is largest for \( \dot{C}_{20} \), contributes only to the excitation of the Chandler wobble and therefore modes due to internal density contrasts are relatively more important. This was first recognized by Yuen & Sabadini (1984) using a simplified boundary condition to model the phase boundary. A more complete treatment of the boundary conditions appropriate for a phase boundary was given by Johnston et al. (1997) and here the effect of assuming those boundary conditions at the 670 and 420 km discontinuities is recalculated.

Johnston et al. (1997) assumed an incompressible earth model for their calculations, but in order to make the comparison with the models in the main body of this paper, this assumption is relaxed and the compressibility of the PREM model (Dziewonski & Anderson 1981) is used. The phase boundaries at 670 and 420 km depth are unlikely to behave completely as isobaric boundaries due to the accumulation of latent heat released by the phase change (O’Connell & Wasserburg 1967; O’Connell 1976; Mareschal & Gangi 1977; Christensen 1985), and the isobaric response factor \( \zeta \) is set to a value of 0.7 for each boundary (\( \zeta = 0 \) corresponds to a chemical or material boundary, \( \zeta = 1 \) corresponds to a fully isobaric boundary). Replacing a material boundary with a phase boundary in an earth model increases the relaxation time of the buoyancy mode and weakens its contribution to radial and geoidal deformation (Johnston et al. 1997).

Fig B1 compares the response factors \( R_C \) and \( R_m \) of the material boundary and phase boundary models as a function of lower-mantle viscosity. The load function is the same as

\begin{center}
\begin{tabular}{c c c}
\hline
R_C (x 10^{-3}) & \hline
\hline
21.0 & 22.0 & 22.5 & 23.0 \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{c c c}
\hline
R_m (x 10^{-3}) & \hline
\hline
21.0 & 22.0 & 22.5 & 23.0 \\
\hline
\end{tabular}
\end{center}

Figure B1. A comparison of the response functions \( R_C \) (left) and \( R_m \) (right) between models with material and phase boundaries at 420 and 670 km depth for a sawtooth load function.

© 1999 RAS, GJI 136, 537–558
the standard load function in Fig. 7, i.e. a sawtooth load with two cycles and initially half-loaded ($f^2_h$) with a 90 kyr loading phase, 10 kyr unloading phase and 7 kyr since the end of the final unloading. As expected, the presence of phase boundaries has a minor effect on $R_C$ because the internal buoyancy modes contribute only a small fraction of the total contribution to $R_C$. The weakening of the contribution of the internal buoyancy modes of relaxation is much more evident in the comparison of $R_m$ between the material and phase boundary models, particularly for $\eta_m < 10^{22}$ Pa s, and the predicted polar wander velocity for the phase boundary model is two-thirds that of the material boundary model if $\eta_m = 10^{21}$ Pa s.

Because $R_C$ is insensitive to the nature of the 420 and 670 km seismic discontinuities, the inference of mantle viscosity from observations of $C_{20}$ as shown in Fig. 12 is unaffected by the assumption of the material boundary conditions at those boundaries. The prediction of polar wander speed, however, is reduced by up to 20 per cent depending on the assumed present rate of sea level change and inferred lower-mantle viscosity. If the present rate of sea level rise is 0.5 mm yr$^{-1}$ and the entire polar wander signal can be explained by postglacial uplift and present melting, then the present polar wander velocity limits the viscosity inference to the lower half of the range inferred from $C_{20}$ (compared with the upper half if a material boundary is assumed).