DEGLACIATION RELATED FEATURES OF THE EARTH'S GRAVITY FIELD

S.M. NAKIBOGLU * and K. LAMBECK

Research School of Earth Sciences, Australian National University, Canberra, A.C.T. 2600 (Australia)

(Received April 1, 1980; accepted June 27, 1980)

ABSTRACT


The Earth's gravity field is partly a consequence of the delayed rebound of previously glaciated regions and this is most readily seen in the negative anomalies over the Hudson Bay, Gulf of Bothnia and the Central Siberian Plateau. Globally the relation between deglaciation and gravity is complicated by the redistribution of the melted ice over the oceans and the deglaciation effects on gravity extend far beyond the boundaries of the former ice loads. In this paper we have computed the contribution to the observed potential of the redistribution of mass associated with the Late Pleistocene deglaciation and found that this is not an insignificant amount. In particular, this contribution exceeds that arising from the Earth's isostatically compensated topography. The observed potential has been corrected for both effects but while details in the geoid or gravity anomaly representation change, the power spectrum is not significantly modified. In particular, the decay of the corrected spectrum remains very similar to that observed.

INTRODUCTION

A perusal of recent solutions of the Earth's gravity field (e.g. Lerch et al., 1979) leads to the suggestion that areas subjected to glaciation in the Late Pleistocene now correspond to negative gravity anomalies. Most notable is the broad negative over the Hudson Bay area of Canada but smaller negative anomalies also occur over the Gulf of Bothnia and over the Central Siberian Plateau. Together with uplift data, these gravity anomalies have been used to estimate the viscosity of the upper mantle (see, for example, Vening Meinesz (1937) for Fennoscandia, and Innes (1960) for the Canadian Shield). Globally, any relationship between gravity and areas of past glaciation is unclear.
for several reasons. These include (1) much of the relaxation has already occurred and the residual gravity anomalies are relatively small, (2) because the departed ice mass has resulted in a global redistribution of surface mass — the concurrent sea-level changes reaching up to 100 m over large areas of the Pacific Ocean (Farrell and Clark, 1976) — the deglaciation effects on gravity extend beyond the boundaries of the immediate ice loads, and (3) other, more dominant factors contribute to the anomalous gravity field. To single out any one feature for special treatment has obvious drawbacks; in particular, many of the world's large shield areas are associated with negative gravity anomalies (Kaula, 1972) but only some of these have also been subject to major ice loading in recent times.

Despite these difficulties there have been several attempts at relating the gravity field to the after-effects of deglaciation. Wang (1966) suggests that the non-hydrostatic flattening and the third degree zonal harmonic of the geopotential could be explained in this way. O'Connell (1971) ruled this out and he also concluded that there was no significant correlation between the ice load and global gravity although he did not consider the effect of the redistribution of the ice mass over the oceans. More recently, Khan and O'Keefe (1974) suggested that the large Antarctic negative anomaly, centered over the Ross Sea, could be related to deglaciation effects although Kaula (1972) had already ruled this possibility out since it would have led to a rise in sea level much greater than has actually been observed.

Ongoing viscoelastic response to deglaciation is, however, still evident in the second degree harmonics of the geopotential as is witnessed by the secular non-tidal acceleration and polar wander of the Earth (Nakiboglu and Lambeck, 1980; referred to hereafter as Paper 1). Because of this it is of interest to re-investigate the contribution that deglaciation effects have made to the present gravity field not because we consider that these contributions are the dominant causes of gravity anomalies, but rather, to "correct" the gravity solutions for this effect.

To do this we have modelled the Earth as a viscoelastic Maxwell body, ignoring thereby all evidence that suggests that a non-linear rheology may be more appropriate, in order to preserve a mathematically simple solution. We justify this choice by the fact that it seems to work in other deglaciation problems (e.g. Peltier et al., 1978; Paper 1) and that we are mainly interested in order of magnitude effects. To solve the problem we first consider the ice—water exchange on a rigid Earth and investigate the correlation of this surface load potential with the observed potential. Next we evaluate the deformation of the Earth under this load using the Love operator approach and evaluate the resulting gravity and geoid changes.

REDISTRIBUTION OF ICE AND WATER LOAD DUE TO DEGLACIATION

The temporal and spatial distribution of the Laurentide and Fennoscandia ice sheets during the Late Pleistocene has most recently been computed by
Peltier and Andrews (1976) and we use here the $10^\circ \times 10^\circ$ area means discussed in Paper 1. The ice melting history is modelled as a linear function of time. The start of deglaciation is taken at $t = 0$ and occurred 18,000 years before the present. The epoch at which melting was completed occurred at about 6000 years before the present. The thickness $\xi_i$ of the ice at a point with colatitude $\theta$, longitude $\lambda$ and at time $t$ is:

$$\xi_i(\theta, \lambda; t) = \xi_i(\theta, \lambda) t_o \left[ 1 - H(t_o - t) + \frac{t}{t_o} H(t_o - t) \right]$$

(1)

where $\xi_i$ is the constant deglaciation rate in the interval $0 \leq t \leq t_o$. The Heaviside unit step function is defined as:

$$H(t_o - t) = \begin{cases} 1 & 0 < t < t_o \\ 0 & t > t_o \\ \frac{1}{2} & t = t_o \end{cases}$$

The resulting change in sea level on a rigid Earth can be computed by observing that the static sea surface remains an equipotential and that the total ice and ocean mass is conserved. The problem leads to an integral equation (Farrell and Clark, 1976) which admits of an approximate solution, as given in Paper 1:

$$\xi_o(\theta, \lambda; t) = \frac{\rho_i}{4\pi \rho_w a_{00}} \int_{\Omega_i} \xi_i \, d\Omega$$

$$+ \frac{3\rho_i}{4\pi \rho_e} \left( \int_{\Omega_i} \frac{\xi_i}{\sin(\psi/2)} \, d\Omega - \left\langle \int_{\Omega_i} \frac{\xi_i}{\sin(\psi/2)} \, d\Omega \right\rangle \right)$$

$$+ \left( \frac{1}{4\pi a_{00}} \int_{\Omega_i} \xi_i \, d\Omega \right) \left( \int_{\Omega_o} \frac{d\Omega}{\sin(\psi/2)} - \left\langle \int_{\Omega_o} \frac{d\Omega}{\sin(\psi/2)} \right\rangle \right)$$

(2)

In this equation:

$\xi_o(\theta, \lambda; t)$ is the change in ocean height on the rigid Earth,
$a_{00}$ is the zero degree harmonic coefficient of the ocean function,
$\rho_i, \rho_w, \rho_e$ are the densities of ice, water and the Earth, respectively.
$\psi$ is the geocentric angle between $(\theta, \lambda)$ and the moving point at the area element $d\Omega(\theta', \lambda')$ on a unit sphere,
$d\Omega = \sin\theta' \, d\theta' \, d\lambda'$,
$\Omega_o$ is the ocean area,
$\Omega_i$ is the ice area,
$\langle \rangle$ indicates the value of a quantity averaged over the oceans.

With eqs. 1 and 2 and with the $10^\circ \times 10^\circ$ average yearly changes of ice thickness based on the Peltier and Andrews compilation the average yearly changes of sea level have been computed for $10^\circ \times 10^\circ$ ocean areas (see Fig. 2 of Paper 1). The height of the global load at a given point can be written
similarly to (1) as:

\[ \xi(\theta, \lambda, t) = \xi(\theta, \lambda) t_o \left[ 1 - H(t_o - t) + \frac{t}{t_o} H(t_o - t) \right] \]  

where

\[ \xi = C_o \xi_o + C_i \xi_i. \]

\( C_o \) and \( C_i \) are the ocean and ice functions. The former is defined as:

\[ C_o = \begin{cases} 
1 & \text{on the oceans} \\
0 & \text{elsewhere} 
\end{cases} \]

and the latter is similarly defined for the ice areas. The gravitational potential of this load on a rigid Earth can be represented in a spherical harmonic expansion as:

\[ \phi(r, \theta, \lambda) = \frac{GM}{R} \sum_{i,n,m} \left( \frac{R}{r} \right)^{n+1} A'_{inm} T(t) \bar{R}_{inm}(\theta, \lambda) \]  

where \( G \) is the gravitational constant and \( M \) the mass of the Earth of mean radius \( R \). The \( \bar{R}_{inm}(\theta, \lambda) = \mathbb{F}_{inm}(\cos \theta) \) \((\delta_{11} \cos m\lambda + \delta_{12} \sin m\lambda)\) are normalized surface spherical harmonics and:

\[ A'_{inm}(t) = \frac{1}{2n + 1} \cdot \frac{3\rho_w}{4\pi R \rho_e} \int_{\Omega = \Omega_i + \Omega_o} \xi(\theta, \lambda) \bar{R}_{inm}(\theta, \lambda) \, d\Omega \]  

with:

\[ T(t) = 1 - H(t_o - t) + \frac{t}{t_o} H(t_o - t) \]

The coefficients \( A'_{inm} \) have been computed up to degree and order 16 by numerical integration of (5) and the combined ice and water height data used in Paper 1.

CORRELATION OF THE LOAD POTENTIAL WITH THE OBSERVED GEO-POTENTIAL

Before proceeding with the study of the response of the Earth to this load we will investigate, in a preliminary manner, the correlation between the potential \( \phi \) and the potential of the presently observed gravity field. This should indicate whether the present field contains significant signatures of past deglaciation and, if it does, at what wavelengths.

The non-hydrostatic part of the exterior gravitational potential of the Earth is:

\[ U = \frac{GM}{R} \sum_{i=1}^{2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^{n+1} C^*_{inm} \bar{R}_{inm}(\theta, \lambda) \]
In this expansion:

\[ C_{inm} = (\tilde{C}_{nm} - \delta_{0m} \tilde{J}_{n0}) \delta_{i1} + \tilde{S}_{nm} \delta_{i2} \]

where \( \tilde{C}_{nm} \) and the \( \tilde{S}_{nm} \) are the observed normalized coefficients of the geopotential. The \( \tilde{J}_{n0} \) are the normalized zonal coefficients describing the hydrostatic equilibrium state. That is \( \tilde{J}_{n0} = -480.5163 \cdot 10^{-3}, \tilde{J}_{40} = 1.212 \cdot 10^{-6}, \tilde{J}_{2n} < O(\tilde{J}_{2}^{2}) \) for \( n > 2 \).

A measure of the correlation between \( U \) and \( \phi \) is given by the degree correlation coefficients:

\[
\beta_n = \frac{\sum_{i} \sum_{m} C_{inm} A_{inm}}{V_n^2(\phi) V_n^2(U)}
\]

where \( V_n^2(\phi) \) represents the degree variances. For example:

\[ V_n^2(U) = \sum_{i} \sum_{m} (C_{inm})^2 \]

Figure 1 illustrates the correlation coefficient together with the corresponding correlation between \( U \) and the equivalent rock topography of Balmino et al. (1973). At first glance it would seem from this result that the gravitational potential carries some signatures of past glaciation at wavelengths corresponding to \( 5 < n < 11 \). However, this is mainly a consequence of the fact that the deglaciation process emphasizes the ocean—continent distribution by removing mass from continental areas and distributing it over the oceans; any phenomena involving mass rearrangements which follow the ocean—continent distribution can be expected to correlate partly with the gravity field. The correlation between gravity and the ocean—continent distribution is partly a consequence of the isostatic compensation of the continents (Lambeck, 1976) but it is also a consequence of plate boundaries—particularly the subduction zones which roughly coincide with continental margins—having pronounced gravity signals. While this latter contribution will undoubtedly be the dominant one, it demonstrates the potential importance of removing the deglaciation effects—and for that matter any crustal effects—from the observed gravity before the observed gravity is interpreted quantitatively in terms of mantle convection.

Deformations characterized by different wavelengths will decay at different rates and short wavelengths in the surface potential can be expected to result in a better correlation with the observed potential than the long wavelengths. This is illustrated in Fig. 2 where the ratio \( (1 + k_n^2) \phi_n/\phi_n \) is plotted as a function of degree. The \( k_n \) are the Love numbers introduced below. For a viscoelastic Earth immediately upon the removal of the load \( k_n = k_n^e \) (the elastic Love numbers). For the present date \( k_n = k_n^v \), the viscoelastic Love number evaluated at \( t = 18,000 \) yrs (see p. 294). Such trends in the correlation are not, however, apparent in Fig. 1.
Fig. 1. Degree correlation coefficients of the geopotential model of Lerch et al. (1979) with \((\alpha)\) the ice and water load, and \((\beta)\) the equivalent rock topography.

Fig. 2. The ratio \((1 + k_n) \phi_n/\phi_n\) where \(\phi_n\) is the load potential. For an elastic response \(k_n = k_{n}^{e}\), the elastic Love number (curve 1). For a viscoelastic response \(k_n = k_{n}^{v}\) (curve 2). The viscoelastic Love numbers correspond to the epoch 18,000 yrs in the ice-load history of eq. 1. The amount of viscous relaxation that has occurred in this interval is indicated by curve 3, and indicates that long wavelengths have relaxed more than short wavelengths.

VISCOELASTIC RESPONSE OF THE EARTH

A convenient way of representing viscoelastic deformations and the resulting perturbations in the gravitational potential is by the Love number formalism. The stress–strain relations for a visco-elastic, homogeneous and isotropic volume, initially in hydrostatic equilibrium, are:

\[
\delta_{ij} + \frac{\mu}{\eta}(\sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}) = \lambda \dot{\varepsilon}_{kk} \delta_{ij} + 2\mu \ddot{\varepsilon}_{ij}
\]  

where \(\sigma_{ij}\) and \(\varepsilon_{ij}\) are the stress and strain tensors, \(\mu\) is the rigidity, \(\eta\) the Newtonian viscosity and \(\lambda\) is the second Lamé constant. Equation 9 can be put into a form analogous to the elastic stress–strain relations using the correspondence principle, (e.g. Bland, 1960), or:

\[
\sigma_{ij} = \hat{\lambda} \varepsilon_{kk} \delta_{ij} + 2\hat{\mu} \varepsilon_{ij}
\]

where \(\hat{\lambda}\) and \(\hat{\mu}\) are Lamé operators defined by:

\[
\hat{\lambda} = \frac{\lambda D + \tau^{-1}(\lambda + \frac{2}{3}\mu)}{D + \tau^{-1}}
\]

\[
\hat{\mu} = \frac{\mu D}{D + \tau^{-1}}
\]
$D$ is the differential operator $d/dt$ (Jeffreys and Jeffreys, 1966) and $\tau = \eta/\mu$ is the relaxation time. Once the elastic problem is solved, the viscoelastic solution follows by replacing $\lambda$ and $\mu$ with $\hat{\lambda}$ and $\hat{\mu}$. For a homogeneous isotropic incompressible elastic sphere, the load Love numbers are (Munk and Mac Donald, 1960):

$$k_n^e = -\frac{1}{1 + N\hat{\mu}}, \quad h_n^e \approx \frac{2}{3}(n + \frac{1}{2}) k_n^e$$

(12)

with:

$$N = 2\left(2n + 4 + \frac{3}{n}\right)/19$$

and:

$$\hat{\mu} = 19\mu/2\rho_\star gR.$$  

For the viscoelastic Earth the response to a potential $\phi_n$ will be (c.f. Peltier, 1974)

$$\Delta \phi_n = \left[1 + k_n^v(D)\right] \phi_n$$

(13)

and:

$$u_n = h_n^v(D) \phi_n/g$$

where $\Delta \phi_n$ is the potential due to the load and the Earth's deformation and $u_n$ is the radial deformation of the planet. With the correspondence principle:

$$k_n^v = -\frac{1}{1 + N\mu^*}, \quad h_n^v = \frac{2}{3}(n + \frac{1}{2}) k_n^v$$

(14)

where:

$$\mu^* = 19\hat{\mu}/2\rho_\star gR.$$  

Combining eqs. 12 and 14 and with the definition (11):

$$k_n^v = k_n^e \left(1 + \frac{1 + k_n^e}{\tau} \frac{1}{D - k_n^e/\tau}\right)$$

(15)

$$h_n^v = h_n^e \left(1 + \frac{1 + k_n^e}{\tau} \frac{1}{D - k_n^e/\tau}\right)$$

Substituting these viscous Love numbers into eqs. 13 gives the time-dependent response of the body. The problem with using the definitions (12) for an elastic Earth is that the elastic response at all wavelengths is determined by only a single parameter $\hat{\mu}$. Hence in (15) we will use not (12) but values computed for a layered Earth with the density and elastic moduli distribution of the model given by Dziewonski et al. (1975).

Table I summarizes the results, based on the above-mentioned Earth model.
### TABLE I
Elastic load Love numbers $k_n^e$ and $h_n^e$ for the Earth model of Dziewonski et al. (1975)

<table>
<thead>
<tr>
<th>n</th>
<th>$-k_n^e$</th>
<th>$-h_n^e$</th>
<th>n</th>
<th>$-k_n^e$</th>
<th>$-h_n^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.290</td>
<td>9</td>
<td>0.074</td>
<td>1.351</td>
</tr>
<tr>
<td>2</td>
<td>0.309</td>
<td>0.993</td>
<td>10</td>
<td>0.071</td>
<td>1.421</td>
</tr>
<tr>
<td>3</td>
<td>0.198</td>
<td>1.050</td>
<td>11</td>
<td>0.068</td>
<td>1.488</td>
</tr>
<tr>
<td>4</td>
<td>0.135</td>
<td>1.051</td>
<td>12</td>
<td>0.065</td>
<td>1.552</td>
</tr>
<tr>
<td>5</td>
<td>0.106</td>
<td>1.084</td>
<td>13</td>
<td>0.063</td>
<td>1.613</td>
</tr>
<tr>
<td>6</td>
<td>0.092</td>
<td>1.141</td>
<td>14</td>
<td>0.062</td>
<td>1.671</td>
</tr>
<tr>
<td>7</td>
<td>0.083</td>
<td>1.210</td>
<td>15</td>
<td>0.060</td>
<td>1.727</td>
</tr>
<tr>
<td>8</td>
<td>0.078</td>
<td>1.281</td>
<td>16</td>
<td>0.058</td>
<td>1.780</td>
</tr>
</tbody>
</table>

and the formalism of the free oscillation equations as developed by Altermann et al. (1959). The resulting viscoelastic Love numbers (15) represent the behaviour of the layered Earth sufficiently well (see Appendix). This is so especially if the viscosity is high and $t$ is of the order $\tau$ or longer. Peltier's (1974; 1976) viscoelastic Love number histories also demonstrate the exponential behaviour with a rate of relaxation proportional to $k_n^e$ for viscosities of $10^{23}$ poise.

**EFFECTS OF DEGLACIATION ON GRAVITY AND ON GEOID HEIGHT**

With respect to the centre of mass of the earth, the change in the geoid height due to the combined effect of the load potential and the Earth's deformation is given by:

$$\delta N(D) = \sum_n (1 + k_n^e) \frac{\phi_n}{g}$$

Substituting (4), (6) and (15) into this expression gives:

$$\delta N(D) = R \sum_i \sum_n \sum_m \frac{\tau D (1 + k_n^e)}{\tau D - k_n^e} \cdot A'_{inm} T(t) \bar{R}_{inm}$$

and integrating over time:

$$\delta N(t) = R \sum_i \sum_n \sum_m N_{inm}(t) \bar{R}_{inm}(\theta, \lambda)$$

where:

$$N_{inm}(t) = A'_{inm} (1 + k_n^e) \frac{\tau}{t_0 k_n^e} \left[ \exp \left( k_n^e \frac{t - t_0}{\tau} \right) - \exp \left( k_n^e \frac{t}{\tau} \right) \right]$$

Figures 3a and 3b illustrate the present geoid with reference to the hydrostatic figure before and after the removal of the deglaciation effects omitting
Fig. 3. a. The observed geoid with reference to the hydrostatic figure including harmonics $2 \leq n \leq 12$. The contour interval is 10 m. b. The geoid corrected for deglaciation effects.

the first degree harmonics and assuming a uniform mantle viscosity of $10^{23}$ poise (Paper 1). The only appreciable difference between the two geoids occurs over the immediate areas of glaciation and the change of sign of $N$ over the Hudson Bay region may be suggestive of a mantle viscosity that is somewhat too high. Elsewhere geoid heights have changed from a few meters to more than 10 m in the northwest Pacific Ocean. The negative anomalies over the Central Siberian Plateau and Antarctica have changed little since these areas were not included in the ice-load history.

The non-hydrostatic potential of the Earth, after removal of the estimated
glaciation effects, is:

\[ V'(D) = \frac{GM}{R} \sum_i \sum_n \sum_m (\frac{R}{r})^{n+1} B_{inm}(D) \bar{R}_{inm}(\theta, \lambda) \]  \hspace{1cm} (17)

where:

\[ B_{inm}(D) = C_{inm}^i - [1 + k_n^i(D)] A_{inm} \]

Some low degree harmonic coefficients of the deglaciation potential \( \delta V \) have been evaluated for the present, that is \( t = 18,000 \) yrs. They are compared in Table II, for selected degree and order with the observed values and with the potential coefficients of the Earth's isostatically compensated topography, namely:

\[ h_{ilm} = \frac{3\rho_c}{\rho_c R} \sum_i \sum_m \frac{1}{2l+1} \left[ 1 - \left( \frac{R - D}{R} \right)^l \right] h_{ilm} R_{inm}(\theta, \lambda) \]  \hspace{1cm} (18)

where \( h_{ilm} \) are the normalized coefficients in the spherical harmonic expansion of the equivalent rock topography, \( \rho_c \) is the density of the crust and \( D \) the average crustal thickness.

The first degree harmonics in the deglaciation potential reflect the shift that occurred in the position of the center of mass of the Earth. Excluding the ice load itself, the shift is \(-4, 6\) and \(-20\) m in \( x, y \) and \( z \), respectively, and is predominantly southwards. The deglaciation contribution to the non-hydrostatic bulge is about 20% of the total and reduces the discrepancy between the observed and hydrostatic values. The equatorial bulge is reduced

**TABLE II**

Some coefficients of observed deglaciation and isostatic gravitational potentials of the Earth (in units of \( 10^{-9} \))

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>Grav. potential ( a )</th>
<th>Deglaciation potential ( b )</th>
<th>Isostatic potential ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{1nm} )</td>
<td>( C_{2nm} )</td>
<td>( (1 + k^t_n) A_{1nm} )</td>
<td>( (1 + k^t_n) A_{2nm} )</td>
<td>( H_{1nm} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( -3.06 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( -0.58 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( -3.65 )</td>
<td>( -0.72 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( -0.18 )</td>
<td>( 0.56 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( 2.43 )</td>
<td>( -1.40 )</td>
<td>( 0.16 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( 0.96 )</td>
<td>( -0.31 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>( -0.67 )</td>
<td>( -0.04 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>( 0.07 )</td>
<td>( 0.26 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>( -0.15 )</td>
<td>( 0.25 )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>( 0.09 )</td>
<td>( 0.24 )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>( 0.05 )</td>
<td>( 0.13 )</td>
<td></td>
</tr>
</tbody>
</table>

*a From Lerch et al. (1979). b At present epoch. c Equation 18 with \( D = 30 \) km.
by about 7%. The third degree zonal harmonic describing the equatorial asymmetry is increased when the deglaciation correction is applied as could be anticipated from heuristic consideration. This harmonic could only be reduced if there has been a substantial deglaciation of Antarctica but this is incompatible with the sea-level changes that have been observed since the Late Pleistocene. If the most recent major Antarctic deglaciation occurred about $4 \times 10^6$ years ago, as is suggested by Hayes et al. (1973), then the size of the ice sheet would have had to be very much larger than it is now, or the upper mantle viscosity would have to be significantly greater than $10^{23}$ poise. Either requirement is implausible on the basis of other observational evidence.

A comparison of normalized power spectra is instructive. The normalized deglaciation potential spectrum is:

$$(2n + 1)^2(1 + k_n^2) \sum_i \sum_m A_{inm}^2$$

and this can be compared with the observed potential spectrum and with the isostatic potential spectrum that follows from (18). The latter is evaluated from $D = 30$ km and is further discussed by Lambeck (1976). Figure 4 summarizes the results together with the normalized spectrum of the "corrected"
potential, that is:

\[(2n + 1)^2 \sum_i \sum_m \left[ C_{inm}^e - (1 + k_n^e) A_{inm} - H_{inm} \right]^2 \]

For harmonics of degree \( n \leq 10 \) the deglaciation potential significantly exceeds the isostatic potential but for \( n \leq 11 \) the two are of very comparable magnitude and the spectrum of the total correction approaches the observed potential at \( n \approx 14 \). In general, the correction for \( n \geq 12 \) are of a similar magnitude to the differences between recent solutions of the geopotential, (see, for example, the comparisons in Lambeck, 1979). The "corrected" geopotential spectrum differs only very little from the observed spectrum and the decay of the spectrum with degree is not significantly modified. In particular, there is no suggestion that the corrected normalized spectrum can be represented by two linear functions as could be the case if the dominant contribution to gravity resulted from the deformations of the boundaries of a convecting layer. Instead, the "corrected" spectrum is compatible with the randomly distributed density anomaly model discussed by Lambeck (1976).

ACKNOWLEDGEMENT

J.B. Merriam has provided the computer program used in the calculation of the elastic and viscoelastic load Love numbers for realistic Earth models. Several discussions with him have also been helpful.

Appendix: THE VISCOELASTIC LOAD LOVE NUMBERS FOR IMPULSE RESPONSE

The load Love numbers for the quasi-homogeneous Earth given in eq. 15 can be compared with the ones corresponding to the real Earth in order to assess the approximation involved in these simple relations. Since the Laplace transform of the impulse response Love numbers can be obtained with relative ease for the real Earth, we will base our comparison on these quantities.

Let us first consider the relaxation of the quasi-homogeneous Earth after an impulse load \( \delta(t) \) is applied at the surface. The viscoelastic load Love numbers for this loading can be computed from eq. 15 as:

\[ k_n^v \delta(t) = k_n^e \delta(t) + k_n^e \frac{(1 + k_n^e)}{\tau} \exp\left(k_n^e \frac{t}{\tau}\right) \]

\[ h_n^v \delta(t) = h_n^e \delta(t) + h_n^e \frac{(1 + k_n^e)}{\tau} \exp\left(k_n^e \frac{t}{\tau}\right) \]

and their Laplace transform are:

\[ K_n(s) = k_n^e \frac{1 + s\tau}{s\tau - k_n^e} \]
Fig. 5. Laplace transforms of the impulse response Love numbers for (a) the quasi-homogeneous Earth, and (b) for a realistic Earth model.
\[ H_n(s) = h_n^e \frac{1 + s\tau}{s\tau} k_n^e \]

where \( s \) is the Laplace transform variable with a dimension of frequency. The preceding expressions are nothing but eq. 15 in which the differential operator \( D \) is replaced by \( s \).

For the layered viscoelastic Earth the Laplace transform of the load Love numbers for impulse loading can be determined readily using the procedure outlined by Peltier (1974). The values of \( K_n(s) \) and \( H_n(s) \) have been determined for selected values of \( s \) by numerically integrating the six well-known ordinary differential equations pertaining to the free oscillations of elastic layered Earth (Altermann et al., 1959). The only difference with the elastic solution is that the Lamé's parameters are replaced by their transforms obtained from eq. 11, i.e.:

\[
\lambda(s) = \frac{\lambda s + \tau^{-1}(\lambda + 2/3\mu)}{s + \tau^{-1}} \\
\mu(s) = \frac{\mu s}{s + \tau^{-1}}
\]

(A.3)

The density and the elastic constants for the Earth are taken from the parametric Earth model of Dziewonski et al. (1975). The viscosity within the Earth is assumed to be \( \eta = 10^{22} \) poise in upper mantle down to a depth of 670 km and \( \eta = 10^{24} \) poise in the lower mantle.

Figures 5a and 5b show the values of \( K_n(s) \) and \( H_n(s) \) for \( 2 \leq n \leq 16 \) for the quasi-homogeneous model and the layered Earth, respectively. The quasi-homogeneous model obviously yields the exact values for large \( s \). Moreover, the Love numbers \( H_n(s) \) are represented well by the simple quasi-homogeneous model at every \( n \) at small and large \( s \). The errors in the simple model are always less than 15%. However, \( K_n(s) \) values for small \( s \) are good to within about 20% for only low degrees, i.e. \( n \leq 6 \).

In view of the uncertainties involved with the late Pleistocene ice thickness and melting history, the simple model is considered to be adequate for global studies and can yield an acceptable approximation for the viscoelastic response of the Earth at long wave-lengths to excitations at intermediate frequency range.

REFERENCES


