COMMENTS ON THERMAL ISOSTASY

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ABSTRACT

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The deformation of the lithosphere due to temperature anomalies caused by a heat source located below or within the layer is usually modelled as one of Pratt local isostasy. A more appropriate model is one of rheologically layered lithosphere comprising of a stiff viscoelastic or elastic layer overlying a weaker viscoelastic layer. The surface deformations are a result of not only the perturbations in body forces due to density changes, but thermal bending moments. In geophysically realistic situations the former contribution dominates. Pratt isostasy is attained if the stresses in the entire lithosphere are allowed to relax and this end state is not contingent upon the lithosphere being confined against horizontal deformation. In a rheologically layered lithosphere, even though the non-isostatic thermal stresses persist in the upper layer, the surface deformations are indistinguishable from that of local isostasy if the horizontal dimension of the heat source exceeds about three times the effective elastic thickness of the lithosphere.

INTRODUCTION

The isothermal response of oceanic lithosphere to surface loading has received considerable attention in the geophysical literature (e.g. Vening Meinesz, 1941; Walcott, 1970; Watts et al., 1980; Lambeck and Nakiboglu, 1980, 1981). The response to thermal loads within the layer has received less attention, studies being restricted to the ocean-ridge models (e.g. Sclater and Francheteau, 1970) and to the mid-ocean swells (e.g. Detrick and Crough, 1978; Sandwell, 1982). For the ocean ridge models the assumption of local isostasy is usually made, while the mid-ocean swell has been discussed in terms of either local isostasy or regional isostasy. Little attention has generally been given to the thermal stress state of this latter loading problem.

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The thermal-isostasy problem is one of finding the deformation and stress-state of a rheologically layered lithosphere that has been subjected to an anomalous transient or steady-state temperature field. The source of this anomalous temperature field may be a heat source immediately below the lithosphere and one complication that arises immediately is that the layer may be in motion relative to the heat source. If the heat source is a mantle plume it may also result in dynamic forces acting on the base of the plate (e.g. McKenzie, 1977; Watts, 1976). We do not consider this latter complexity here and limit the discussion to the problem of finding a solution for the deformation and stress state of a lithosphere subjected to an anomalous temperature field only. The problem differs from the isothermal surface loading one in several important respects. In the first instance the density field is perturbed throughout the heated region of the plate causing changes in body forces. Secondly, the anomalous temperature field creates in-plane forces as well as bending moments. Unlike the isothermal loading effects the thermal effects vary with time and also with depth even for an elastic lithosphere because the anomalous temperature field is depth dependent and the time required to attain a steady state can be long, of the order $(1-5)10^7$ years.

The paper is restricted to the discussion of two fundamental aspects of thermal isostasy. First we establish the underlying assumptions of Pratt isostasy and its stress state. Hence we clarify the confusion found in the geophysical literature on whether a linear (α) or volumetric (3α) expansion coefficient should be used in modelling the topography of ocean ridges. The usual interpretation of Pratt isostasy, starting with Pratt (1859), is that the lithosphere has been allowed to expand freely in the vertical direction only and that the volumetric coefficient of thermal expansion is used (e.g. Sclater and Francheteau, 1970; Le Pichon et al., 1973; Davis and Lister, 1974). In this interpretation the Pratt model's analogy of rising dough is valid only if the dough is contained laterally in a baking tin, but now horizontal stresses can be substantial and this is inconsistent with the isostatic stress state in which stress-differences are minimized (e.g. Jeffreys, 1970). Morgan (1975) used the correct underlying assumption that all horizontal stresses must vanish and used 3α. Le Pichon et al. (1973, p. 62) argue that α should be used but, in order to match theory with observations, continue to use 3α to compute the density change and permit expansion to occur only in the vertical direction. Bottinga and Allegre (1976, p. 21) also comment on this usage but use 3α. Bottinga and Steinmetz (1979, p. 325) use α and a formulation of Pratt isostasy that is consistent with the model discussed below.

The second point relates to the depth dependence of the rheology of the crust of lithosphere. Studies of isothermal loading of oceanic lithosphere under even young volcanic loads is not greater than about 15–20 km (e.g. Walcott, 1970; Watts et al., 1980; Lambeck, 1981). The thermal thickness of the oceanic lithosphere is generally much thicker, of the order 70–100 km (e.g. Parsons and Sclater, 1977). It appears, therefore, that the lower lithosphere is sufficiently ductile for load-generated stresses to migrate from the lower parts of the lithosphere to the upper regions in a time period which is approximately equal to the duration of seamount formation or a few million years. Similarly, thermal stresses created in this lower layer by an anomalous heat source will also relax relatively quickly when compared with the upper part of the lithosphere, particularly when the increased temperature in the lower lithosphere due to the heat source causes a further decrease in viscosity. Hence the stresses in this part of the layer can be expected to relax quickly. Thus when examining the state of isostasy of a lithosphere subjected to a heat source within its lower regions, or below it, the conventional Pratt model may not be appropriate since the horizontal stresses would relax rapidly only in the lower regions and thermal expansion may not occur freely at all depths. Now a model of partly regional, partly local isostatic equilibrium is appropriate.

DENSITY CHANGES DUE TO TEMPERATURE ANOMALIES

Consider an elastic, homogeneous, isotropic body whose thermo-mechanical properties are independent of temperature. Let the temperature distrubution within the body be perturbed by $T(x_1, x_2, x_3)$. The resulting thermal stresses and strains satisfy the following equilibrium and Duhamel-Neumann equations (e.g. Boley and Weiner, 1960; Nowacki, 1962; Fung, 1965)

$$\sigma_{ij,j} + f_i = 0, \tag{1a}$$

$$\varepsilon_{ij} = \alpha T \delta_{ij} + \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right), \tag{1b}$$

where σ_{ij} and ε_{ij} are the thermal stress and strain tensors respectively. f_i are the components of the body force per unit volume, δ_{ij} is the Kronecker's delta, μ is the rigidity, and ν is Poisson's ratio. The cubical dilation ε_{jj} follows by contracting equation 1b, namely

$$\varepsilon_{jj} = 3\alpha T + \frac{1-2\nu}{2(1+\nu)\mu}\sigma_{jj},$$

and the change in density relative to its value ρ_0 in the unperturbed state is

$$\rho(x_1, x_2, x_3) = \rho(1 + \varepsilon_{jj})^{-1}.$$

That is

$$\Delta\rho = \rho - \rho_0 \cong -3\alpha T \rho_0 - \rho_0 \frac{1-2\nu}{2(1+\nu)\mu}\sigma_{jj}, \tag{2a}$$

or

$$\Delta\rho = -\rho_0(3\alpha T + \frac{1}{3k}\sigma_{jj}), \tag{2b}$$

where k is the bulk modulus.

The first term of (2b) will generally be the dominant one under lithospheric conditions. The stress in the second term is of the order $3\alpha ET/(1+\nu)$ and the ratio of the first to second term is approximately $2(1-\nu)/(1-2\nu)$ and the first term is about four times the second when $\nu \simeq 0.3$. Hence, irrespective of the horizontal dimensions of the lithosphere, one can usually adopt

$$\Delta\rho \cong -3\alpha T\rho_0. \tag{3}$$

This approximation becomes even more accurate if the lithosphere is allowed to relax under thermal stresses as a viscoelastic body because the bulk modulus is time invariant and σ_{jj} decreases with time.

THE NATURE OF PRATT ISOSTASY

If the lithosphere can be approximated as a thin plate and the thermal deformations are small, the three dimensional elasticity problem defined by equations (1) is reduced to a two-dimensional problem of plane strain. A coordinate system is adopted whose origin lies at the upper surface of the plate and whose third axis, z, points downwards. The governing differential equations for the vertical deformations w (positive downwards) are (Pister and Dong, 1959)

$$D\nabla^4 w = \nabla^2\left(\frac{d}{2}N_T - M_T\right) + q + F_{,yy}W_{,xx} - 2F_{,xy}W_{,xy} + F_{,xx},W_{yy}, \tag{4a}$$

$$\nabla^4 F = -(1-\nu)\nabla^2 N_T, \tag{4b}$$

where ∇^4 is the biharmonic differential operator,

$D = \dfrac{\mu d^3}{6(1-\nu)}$ is the flexural rigidity of the plate,

d is the plate thickness,

$N_T = 2\mu\alpha\dfrac{1+\nu}{1-\nu}\displaystyle\int_{z=0}^{d} T dz$ is the thermal stress resultant,

$M_T = 2\mu\alpha\dfrac{1+\nu}{1-\nu}\displaystyle\int_{z=0}^{d} Tz dz$ is the thermal bending moment,

F is the stress function, and

q is the vertical (lateral) load acting on the plate.

We have changed the coordinate notation from x_i to x, y, z with x, y being in the horizontal plane. The notation $(\ )_{,xy}$ denotes the differentiation $\partial^2(\ )/\partial x\,\partial y$. The more commonly used plate equations (e.g. Boley and Weiner, 1960) can be deduced from equation 4a by translating the coordinate system to the mid-plane of the plate, in which case the term containing N_T on the right-hand side of this equation vanishes. In the case of a submerged plate with its density perturbed due to temperature change T, the vertical load can be obtained from the general plate equations including body forces (Fung, 1965) as

$$q \cong -(\rho_m - \rho_w)\, g w - 3a\rho_0\, g \int_0^d T dz. \tag{5}$$

Here ρ_m, ρ_w and ρ_0 are the densities of mantle, water and the unperturbed lithosphere respectively, g is the acceleration of gravity.

If the plate edges are unrestricted so that the mid-plane is free of horizontal stress, and further, if the temperature perturbation at any x, y does not deviate appreciably from a linear thermal gradient across the plate thickness, then the stress resultants are approximately zero, i.e.

$$N_x = F_{,yy} = 2\mu \frac{1+\nu}{1-\nu} \alpha(T d - \int_0^d T dz) \simeq 0$$

$$N_y = F_{,xx} = N_x = 0,$$

$$N_{xy} = -F_{,xy} = 0.$$

Hence the two plate equations 4a,b reduce to a single equation

$$D\nabla^4 w + (\rho_m - \rho_w)\, g w = \nabla^2\left(\frac{d}{2} N_T - M_T\right) - 3a\rho_0\, g \int_{z=0}^d T dz. \tag{6}$$

The deflection computed from this formulation is only valid if the midplane is able to stretch or contract freely under the thermal influences. If the horizontal deformation is restricted, then the stress resultants are

$$N_x = N_y = -N_T,$$

$$N_{xy} = 0,$$

and equations 4a,b become

$$D\nabla^4 w + N_T \nabla^2 w = \nabla^2\left(\frac{d}{2} N_T - M_T\right) + q. \tag{7}$$

Now the vertical load q needs to be computed using equation 2 since the stress tensor trace σ_{jj} may now not be negligible. For a compressible viscoelastic plate, equation 6 becomes, by virtue of the correspondence principle,

$$D\hat{\nabla}^4 \hat{w} + (\rho_m - \rho_w)\, g \hat{w} = -3a\rho_0 g \int_0^d \hat{T}\, dz + \nabla^2\left(\frac{d}{2} \hat{N}_T - \hat{M}_T\right), \tag{8}$$

with

$$\hat{D} = \frac{\hat{\mu} d^3}{6(1-\hat{\nu})},$$

$$\hat{\mu} = \frac{\mu s}{s + 1/\tau},$$

$$\hat{\nu} = \frac{\lambda s + k/\tau}{2[(\lambda + \mu)s + k/\tau]},$$

$$\hat{N}_T = \frac{2\hat{\mu}\alpha(1+\hat{\nu})}{1-\hat{\nu}} \int_0^d \hat{T}\, dz,$$

$$\hat{M}_T = \frac{2\hat{\mu}\alpha(1+\hat{\nu})}{1-\hat{\nu}} \int_0^d \hat{T} z\, dz,$$

s is the Laplace transform variable, τ is the ratio of viscosity η to rigidity μ, and λ is the Lame parameter of the plate. $(\hat{\ })$ indicates the Laplace transform of a function of time. $\hat{D}$, $\hat{\mu}$, and $\hat{\nu}$ are the differential operators which characterize the plates viscoelastic memory of the past loads.

The local isostatic state is attained when $t \gg \tau$ at which time

$$w_{is} = \lim_{t \to \infty} w(t)$$

or

$$w_{is} = \lim_{s \to 0} (sw)$$

Multiplying equation 8 by s and taking the limit as $s \to 0$ we obtain

$$w_{is}(x, y) = -3\alpha \frac{\rho_0}{\rho_m - \rho_w} \int_0^d T^\infty(x, y, z)\, dz. \tag{9}$$

Here T^∞ is the steady-state temperature anomaly in the lithosphere.

Equation 9 is the statement of Pratt isostasy for a submerged plate. It represents the limiting state of a viscoelastic lithosphere which has been able to expand freely in the horizontal direction at all depths. The simple plate equation and consequently the expression for Pratt isostasy is based on the assumption that the vertical deformation is independent of depth. Therefore, equation 9 also gives the deformation at the base of the lithosphere.

REGIONAL ISOSTATIC DEFORMATION OF LAYERED LITHOSPHERE

We consider a model of the lithosphere in which an elastic or stiff viscoelastic layer overlies a weaker viscoelastic layer. The base of the upper layer is taken to define the long-term effective elastic thickness and the base of the lower, less viscous, layer defines the thermal thickness of the lithosphere. The two layers are coupled in both horizontal and vertical directions such that deformations are continuous across the interface of the two layers although there may exist a discontinuity in the horizontal stresses depending on the rigidity contrast between the layers. A more realistic model is one in which the change in rheology with depth is gradual but the present model suffices if the parameters are interpreted as effective or equivalent values.

The theory of isothermal loading of layered elastic-viscoelastic plates is given in Lambeck and Nakiboglu (1981). Combining this development with the theory of thermal deformations of layered elastic plates given in Pister and Dong (1959), one obtains the governing differential equation for the layered elastic-viscoelastic plate under thermal perturbances. The equation can be cast in a form similar to that of a single layer plate (equation 8) as

$$\hat{D}^*\nabla^4\hat{w} + (\rho_m - \rho_w)\, g\hat{w} = \nabla^2\left(\frac{\hat{B}}{\hat{A}}\hat{N}_T - \hat{M}_T\right) - \hat{f}_3, \tag{10}$$

where

$$\hat{A} = 2\left[\frac{\mu_1 d_1}{1-\nu_1} + \frac{\hat{\mu}_2}{1-\hat{\nu}_2}(d_2 - d_1)\right],$$

$$\hat{B} = \frac{\mu_1 d_1^2}{1-\nu_1} + \frac{\hat{\mu}_2}{1-\hat{\nu}_2}(d_2^2 - d_1^2),$$

$$\hat{C} = \frac{2}{3}\left[\frac{\mu_1 d_1^3}{1-\nu_1} + \frac{\hat{\mu}_2(d_2^3 - d_1^3)}{1-\nu_2}\right],$$

$$\hat{D}^* = \hat{C} - \hat{B}^2/\hat{A},$$

$$\hat{M}_T = 2\frac{\mu_1\alpha_1(1+\nu_1)}{1-\nu_1}\int_0^{d_1} z\hat{T}\,dz + 2\frac{\hat{\mu}_2\alpha_2(1+\hat{\nu}_2)}{1-\hat{\nu}_2}\int_{d_1}^{d_2} z\hat{T}\,dz,$$

$$\hat{N}_T = 2\frac{\mu_1\alpha_1(1+\nu_1)}{1-\nu_1}\int_0^{d_1} \hat{T}\,dz + 2\frac{\hat{\mu}_2\alpha_2(1+\hat{\nu}_2)}{1-\hat{\nu}_2}\int_{d_1}^{d_2} z\hat{T}\,dz,$$

$$\hat{f}_3 = -3g\left[\rho_{0_1}\alpha_1\int_0^{d_1}\hat{T}\,dz + \rho_{0_2}\alpha_2\int_{d_1}^{d_2}\hat{T}\,dz\right],$$

In these expressions d_1 and $d_2 - d_1$ are the thicknesses of the elastic and viscoelastic layers respectively. The subscripts 1 and 2 denote the quantities pertaining to the upper (elastic) and lower (viscoelastic) layers respectively. We assume that the density and linear expansion coefficients are the same in both layers and that $\mu_1 = \mu_2(t=0)$ and $\nu_1 = \nu_2(t=0)$. Furthermore, Poisson's ratio is assumed to be sufficiently high in the lithosphere so that its increase in time in the lower parts towards the incompressible limit of 0.5 can be ignored. The viscosity η is sufficiently low so that the stresses in the lower lithosphere can be expected to relax quickly and we need be concerned only with the asymptotic behaviour of the elastic-viscoelastic lithosphere at times sufficiently large compared with the relaxation time constant τ of the lower lithosphere. These approximations should lead to a reasonable first approximation for the solution of the problem.

Multiplying the plate equation 10 by s and taking the limit $s \to 0$, the governing equation for regional isostasy becomes

$$D\nabla^4 w + (\rho_m - \rho_w) g w = 3\alpha\mu_1 d_1^2 \nabla^2 (T_{01}^{\infty} - 2T_{11}^{\infty}) - 3\alpha\rho_0 g d_2 (\gamma T_{01}^{\infty} + T_{02}^{\infty}), \quad (11)$$

where $D = \dfrac{\mu_1 d_1^3}{3}$ and $\gamma = \dfrac{d_1}{d_2}$.

The moments of steady-state anomalous temperature field are defined as

$$T_{ij}^{\infty} = \frac{1}{d_j^{i+1}} \int_{d_{j-1}}^{d_j} z^i T^{\infty}\, dz$$

Defining the flexural parameter l as

$$l = \left[\frac{D}{(\rho_m - \rho_w) g} \right]^{1/4}, \quad (12)$$

and substituting it in equation 11 yields

$$\nabla^4 w + l^{-4} w = \frac{9\alpha}{d_1} \nabla^2 (T_{01}^{\infty} - 2T_{11}) - 3\alpha \frac{\rho_0}{\rho_m - \rho_w} \frac{d_2}{l^4} (\gamma T_{01}^{\infty} + T_{02}^{\infty}) \quad (13)$$

The Green's function G_w for the regional isostatic deformation can be obtained from the solution of

$$\nabla^4 G_w + l^{-4} G_w = \delta(r) \quad (14)$$

where $r = (x^2 + y^2)^{1/2}$ and $\delta(r)$ is the Dirac delta function. Taking the zero order Hankel transform of this equation yields

$$\tilde{G}_w(\xi) = \frac{1}{2\pi} \frac{1}{\xi^4 + l^{-4}}.$$

where $\tilde{G}_w(\xi) = \int_{r=0}^{\infty} G_w(r)\, J_0(\xi r) r\, dr$ is the Hankel transform of $G_w(r)$, ξ is the Hankel transform variable, and J_0 is the zero order Bessel function of the first kind.

The inverse Hankel transform of the preceeding equation results in

$$G_w(r) = -\frac{l^2}{2\pi} \operatorname{kei}\left(\frac{r}{l}\right). \quad (15)$$

Here, kei is the Kelvin-Bessel function of zero order. The general solution of the differential equation 13 follows from the convolution of its right-hand side with the Green's function G_w, or

$$w(r,\theta) = -\frac{9al^2}{2\pi d_1}\int_{r'=0}^{\infty}\int_{\theta'=0}^{2\pi} \operatorname{kei}\left|\frac{\underline{r}-\underline{r}'}{l}\right| \times \nabla^2\left[T_{01}^{\infty}(r',\theta') - 2T_{11}^{\infty}(r',\theta')\right] r'\,dr'\,d\theta' + \frac{3ad_2}{2\pi l^2}\frac{\rho_0}{\rho_m-\rho_w}\int_{r'=0}^{\infty}\int_{\theta'=0}^{2\pi} \operatorname{kei}\left|\frac{\underline{r}-\underline{r}'}{l}\right| \times \left[\gamma T_{01}^{\infty}(r',\theta') + T_{02}^{\infty}(r',\theta')\right] r'\,dr'\,d\theta'.$$

Here r, θ denote the cyclindrical polar coordinates and $\underline{r}$ and $\underline{r}'$ are the position vectors of the fixed and moving points respectively. The first integral on the right-hand side can be further simplified by using Green's first identity and noting that

$$\lim_{r'\to\infty} r'\frac{\partial}{\partial r'}\operatorname{kei}\left|\frac{\underline{r}-\underline{r}'}{l}\right| \to 0,$$

and

$$\nabla^2 \operatorname{kei}\left|\frac{\underline{r}-\underline{r}'}{l}\right| = \frac{1}{l^2}\operatorname{ker}\left|\frac{\underline{r}-\underline{r}'}{l}\right|.$$

with the result

$$w(r,\theta) = -\frac{9a}{2\pi d_1}\int_{r'=0}^{\infty}\int_{\theta'=0}^{2\pi} \operatorname{ker}\left|\frac{\underline{r}-\underline{r}'}{l}\right| \times \left[T_{01}^{\infty}(r',\theta') - 2T_{11}^{\infty}(r',\theta')\right] r'\,dr'\,d\theta' + \frac{3ad_2}{2\pi l^2}\frac{\rho_0}{\rho_m-\rho_w}\int_{r'=0}^{\infty}\int_{\theta'=0}^{2\pi} \operatorname{kei}\left|\frac{\underline{r}-\underline{r}'}{l}\right| \times \left[\gamma T_{01}^{\infty}(r',\theta') + T_{02}^{\infty}(r',\theta')\right] r'\,dr'\,d\theta'. \tag{16}$$

The above integrals cannot be evaluated analytically for realistic temperature fields resulting from heating of the lithosphere from below and a numerical convolution scheme is given in the Appendix.

The solution (16) (or the equivalent solution A1) represent the case where the thermally induced stresses in the lower and more ductile part of the lithosphere have diffused with time such that the region has reached the isostatic stress state. The first integral represents the deformation due to the thermal bending moment created in the elastic layer by the heat source below the lithosphere. Figure 1 illustrates this effect due to a steady-state temperature field generated by an anomalous heat source under a lithospheric plate moving

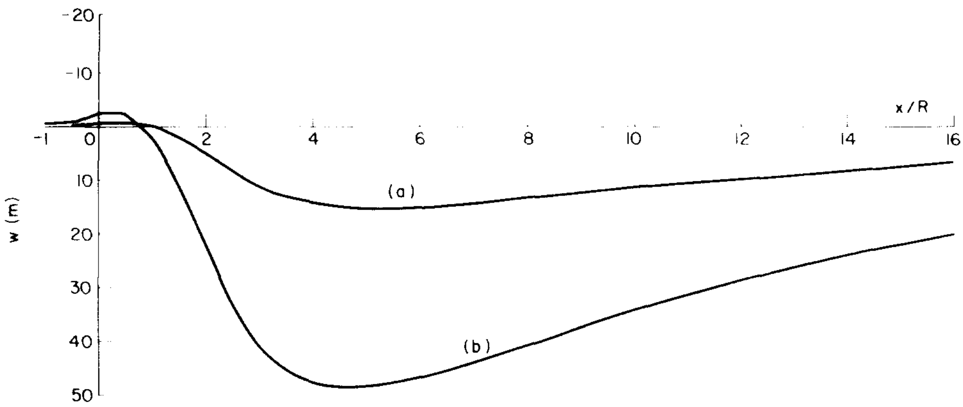

Fig. 1. Surface deformation of a 100 km thick elastic-viscoelastic lithosphere due only to the thermal bending moment generated by a basal heat source with radius $R = 200$ km, intensity $q = 0.21$ Wm^{-2} (5 H.F.U.) and centered at $x = 0$. The lithospheric plate moves in the x direction with a velocity of 2 cmy^{-1} relative to the heat source. Positive deformation indicates subsidence. The curves are for (a) 20 km, and (b) 30 km thick elastic layer.

with a velocity of 2 cmy^{-1} relative to the source. The temperature field is that given in Nakiboglu and Lambeck (1984). The radius R and the intensity q of the heat source are taken to be 200 km and 0.21 Wm^2 (5 H.F.U.) respectively. The thickness of the thermal lithosphere is 100 km and the thickness of the elastic part is varied between 20 and 30 km. As seen in figure 1, a very small uplift develops over the heat source but then subsides rapidly as the plate moves away from the heat source. This subsidence reaches its maximum at a distance of about 4R and then diminishes more gradually further downstream. One interesting observation is that the thicker the elastic layer the larger the deformations because the magnitude of thermal bending moments increases with plate thickness. This behaviour is opposite to the response of the lithosphere to isothermal loads.

The second integral in equation 16 represents the deformation induced by the body forces arising from the change in the lithosphere's weight due to thermal expansion. The influence of this term is illustrated in Figures 2a and 2b. Again the deformation field is carried downstream due to relative plate motion which results in an elongated uplift region some tens of times larger than the areal extend of the heat source. This effect is much greater than that generated by the thermal moments unless the whole lithosphere behaves elastically.

Figures 2a and 2b also include the local isostatic deformation as calculated from equation 10. Regional isostatic deformations are smaller than local isostatic ones only if the heat source is small in areal extend. For a basal heat source of 200 km the local and regional deformations are almost identical if the elastic thickness of the lithosphere is less than 30 km, and the distinction between local and regional compensation models will in general be negligible.

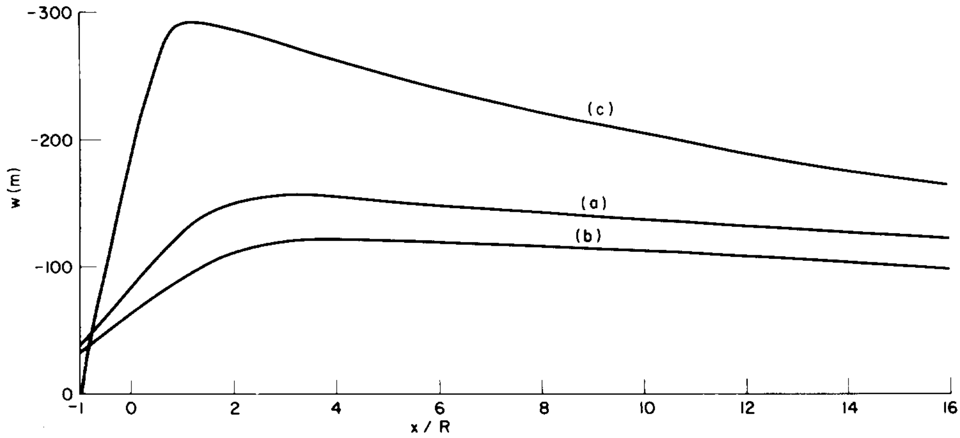

Fig. 2a. Surface deformation of a 100 km thick elastic-viscoelastic lithosphere due to density perturbations generated by a basal heat source with radius $R = 50$ km and intensity $q = 0.21$ Wm^{-2} (5 H.F.U.). The lithospheric plate moves in the x direction with a velocity of 2 cmy^{-1} relative to the heat source. The thickness of the elastic layer is 20 km (curve a), and 30 km (curve b). The local isostatic uplift is also shown (curve c).

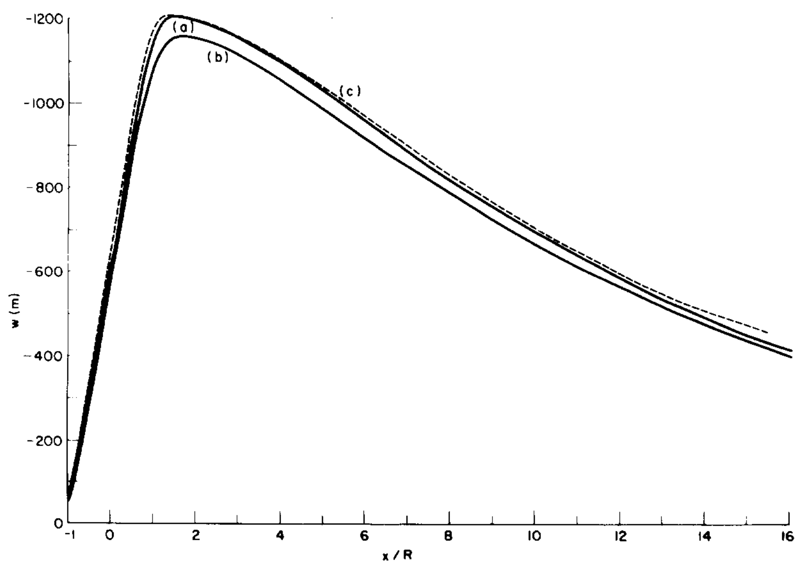

Fig. 2b. Same as in figure 2a but for $R = 200$ km.

DISCUSSION

In the derivation of equations (3) and (9) the plate has been allowed to expand in the horizontal direction and clearly the 3α is not a consequence of constraining the plate in this direction. Thus while equation (9) is consistent with the normal use of the Pratt model, its stress state is different from that frequently implied. In the model the horizontal stresses σ_{11}, σ_{22}, σ_{12}, σ_{31} and σ_{32} are everywhere zero and the deviatoric stresses in any vertical plane within the lithosphere are equal to σ_{33} and zero in the horizontal plane. This stress state is not hydrostatic and further relaxation can occur such that the hydrostatic state is approached.

In the layered model, the deformations caused by thermal bending moments are negligible compared with those caused by the change in the body force although the stress state in this model is substantially different from that of the Pratt model in which the body force is the only contribution to deformation. In particular, the horizontal stresses may be quite significant in the upper layer, a region that is also relatively cold and capable of withstanding such stresses for long intervals of time.

Two recent papers relate directly to the two points of thermal isostasy raised here. Sandwell (1982) modelled the state of isostasy for the mid-ocean swell by considering it as an elastic layer over a plastic lower lithosphere. He solved the problem by loading the elastic plate from below with a pressure term p such that his governing equation is

$$\nabla^2\nabla^2 w + (\rho_m - \rho_w)\, g w = -p.$$

Any thermal stresses generated within the elastic layer are therefore ignored and, as shown above, this is mostly valid insofar as the deformation is concerned. The pressure term p is defined by Sandwell as (his equation A16)

$$p = -3\alpha\rho_0\, g \int_0^d T\, dz$$

where the integral is taken over the entire thickness of the lithosphere. (Note the different sign convention adopted for p and that α, as used here, is the linear coefficient of thermal expansion). But for a plastic medium the pressure term induced by a temperature change is

$$p = 2\,\frac{(1+\nu)}{1-2\nu}\,\mu \left[\frac{\varepsilon_{kk}}{3} - \alpha T\right],$$

and is a function of the ambient anomalous temperature only. Furthermore, if the medium is free to expand, $\Delta = 3\alpha T$ and $p = 0$ and this is also the case in an unconfined viscoelastic medium. Only if the lithosphere is placed in Pratt's baking dish will p be non-zero, i.e.

$$p = -3a\rho_0 \int_0^{d-d_1} T dz$$

where the integral is across the lower layer of the lithosphere. What the derivation leading to (11) or (16) indicates is that the correct interpretation of the isostatic state is not in terms of a basal pressure but in terms of the body force $g\Delta\rho$. Numerically Sandwell's result is very similar to ours when the thermal bending moments are neglected.

The second pertinent paper is by Bills (1983) who has examined the role of the thermal bending moments in the lithosphere modelled on a homogeneous plate. Bills concludes that these bending moments lead to a very significant surface subsidence and that they may play an important role in sedimentary basin formation, for example, by loading the depressions in Figure 1 by sediments. Bills ignores the fact that the lower lithospheric stresses and hence the thermal bending moments are likely to relax more quickly than the conduction of heat upwards towards the stronger regions and that the change in body force in this part of the layer makes a substantial contribution to the surface deformation. Anomalous temperature propagates upwards at time periods of the order (1–5) 10^7 years (Nakiboglu and Lambeck, 1984) and the lower lithosphere will have had time to relax before the near surface and cooler region will have experienced any temperature change. This situation becomes even more pronounced if the lithosphere is moving relative to the heat source.

No attempt is made here to apply this model to geophysical problems for the simple reason that in most situations the simple Pratt model will satisfy the surface observations. The layered model is more pertinent in situations where the thermal state of the lithosphere is perturbed from within or from below, such that the upper layers remain relatively cool and competent. Thus an obvious application is to the Hawaiian swell and similar mid-ocean swells (Detrick and Crough, 1978). These bathymetric features, associated with mid-plate volcanism, have quite similar characteristics to those illustrated in Figures 2a,b. If such features represent passive mechanical surface loads then the associated stresses and the wavelengths of these "loads" are excessive and cannot be supported by the lithosphere. Instead, these loads would stress the mantle down to considerable depths. Much more likely is that these features are of thermal consequence and now they can be supported by the lithosphere. This distinction may not always have been appreciated. Watts (1976), for example, while arguing for a thermal origin of the feature concluded that it could not be supported statically by the lithosphere and that dynamic support, associated with a mantle plume, was essential. While partial dynamic support is not unreasonable (Nakiboglu and Lambeck, 1984; McKenzie, 1977), Watt's argument in reaching this conclusion is incorrect.

APPENDIX

A numerical convolution scheme for evaluating equation 16 can be developed by subdividing the lithosphere into small columns in each of which the thermal moments T_{ij}^{∞} can be taken constant. This is a reasonable approximation in view of the smooth variation of anomalous temperature field (see e.g. Paper 1) provided the subdivision areas are small compared to the lithospheric thickness. Hence dividing the lithosphere into cylincrical columns of radius A one obtains an approximation to (16) as

$$
\begin{aligned}
w(r_i,\theta_i) = &-\frac{9\alpha}{2\pi d_1}\sum_{j=1}^{n}\left[T_{01}^{\infty}(r_j,\theta_j) - 2T_{11}^{\infty}(r_j,\theta_j)\right] \\
&\times \int_{r=0}^{A}\int_{\theta=0}^{2\pi} \operatorname{ker}\left|\frac{\underline{r}_i-(\underline{r}_j+\underline{r})}{l}\right| r\,dr\,d\theta \\
&+\frac{3\alpha d_2}{2\pi l^2}\frac{\rho_0}{\rho_m-\rho_w}\sum_{j=1}^{n}\left[\gamma T_{01}^{\infty}(r_j,\theta_j) + T_{02}^{\infty}(r_j,\theta_j)\right] \\
&\times \int_{r=0}^{A}\int_{\theta=0}^{2\pi} \operatorname{kei}\left|\frac{\underline{r}_i-(\underline{r}_j+\underline{r})}{l}\right| r\,dr\,d\theta.
\end{aligned}
$$

Here (r_j, θ_j) are the coordinates of the centroid of the j^{th} column and n is the total number of columns.

The integrals of ker and kei can be evaluated using the decomposition formulas for these funcions resulting in

$$
\begin{aligned}
w(r_i,\theta_i) = &-\frac{9\alpha}{2\pi d_1}\sum_{j=1}^{n} F_{ij}^{1}\left[T_{01}^{\infty}(r_j,\theta_j) - 2T_{11}^{\infty}(r_j,\theta_j)\right] \\
&+\frac{3\alpha d_2}{2\pi l^2}\frac{\rho_0}{\rho_m-\rho_w}\sum_{j=1}^{n} F_{ij}^{2}\left[\gamma T_{01}^{\infty}(r_j,\theta_j) + T_{02}^{\infty}(r_j,\theta_j)\right],
\end{aligned}
\tag{17}
$$

where

$$
F_{ij}^{1} = 2\pi A l \begin{cases} \operatorname{kei}'\dfrac{A}{l} & \text{if } i=j \\[2ex] \left(\operatorname{ker}\left|\dfrac{\underline{r}_i-\underline{r}_j}{l}\right| \operatorname{bei}'\dfrac{A}{l} + \operatorname{kei}\left|\dfrac{\underline{r}_i-\underline{r}_j}{l}\right| \operatorname{ber}'\dfrac{A}{l}\right) & \text{if } i\neq j \end{cases}
$$

$$
F_{ij}^{2} = 2\pi A l \begin{cases} -\dfrac{l}{A}\left(1+\dfrac{A}{l}\operatorname{ker}'\dfrac{A}{l}\right) & \text{if } i=j \\[2ex] \left(\operatorname{kei}\left|\dfrac{\underline{r}_i-\underline{r}_j}{l}\right| \operatorname{bei}'\dfrac{A}{l} - \operatorname{ker}\left|\dfrac{\underline{r}_i-\underline{r}_j}{l}\right| \operatorname{ber}'\dfrac{A}{l}\right) & \text{if } i\neq j \end{cases}
$$

Here prime denotes the derivatives of the zero-order Kelvin Bessel functions ber, bei, ker and kei.

Both ker and kei diminish rapidly with distance therefore deformation $w(r_i, \theta_i)$ is influenced by the temperature moments and body forces in the neighbourhood of point i and contributions of columns at large distances to the point are negligible.

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